MTH229

Contents

CHAPTER 1

Using Julia as a Calculator

Quick background

Read about this material here: Julia as a calculator.

For the impatient, these questions cover the use of `julia` to replace what a calculator can do.

The common operations on numbers: addition, subtraction, multiplication, division, and powers.

For the most part there is no surprise, once you learn the notations: `+`, `-`, `*`, `/`, and `^`. (Though you may find that copying and pasting minus signs will often cause an error, as only something that looks like a minus sign is pasted in.)

Using `IJulia`, one types the following into a cell and then presses the *run* button (or *shift-enter*):

```
2 + 2
```

```
4
```

The answer follows below the cell.

Here is how one does a slightly more complicated computation:

```
(2 + 3)^4/(5 + 6)
```

```
56.81818181818182
```

As with your calculator, it is very important to use parentheses as appropriate to circumvent the usual order of operations.

The use of the basic families of function: trigonometric, exponential, logarithmic.

On a calculator, there are buttons used to compute various functions. In `julia`, there are *many* pre-defined functions that serve a similar role (and you will see how to define your own). Functions in `julia` have names and are called using parentheses to enclose their argument(s), as with:

```
sin(pi/4), cos(pi/3)
```

```
(0.7071067811865475,0.5000000000000001)
```

(With `IJulia`, when a cell is executed only the last command computed is displayed, the above shows that using a comma to separate commands on the same line can be used to get two or more commands to be displayed.)

Most basic functions in `julia` have easy to guess names, though you will need to learn some differences, such as `log` is for ln and `asin` for $\sin^{-1}$.

the use of memory registers to remember intermediate values.

Rather than have numbered memory registers, it is *easy* to assign a name to a value. For example,

```
x = 42
```

```
42
```

Names can be reassigned (though at times names for functions can not be reassigned to different types of values). For assigning more than one value at once, commas can be used as with the output:

```
a,b,c = 1,2,3
```

```
(1,2,3)
```

Julia, like math, has different number types

Unlike a calculator, but just like math, `julia` has different types of numbers: integers, rational numbers, real numbers, and complex numbers. For the most part the distinction isn't much to worry about, but there are times where one must, such as overflow with integers. (One can only take the factorial of 20 with 64-bit integers, whereas on most calculators a factorial of 69 can be taken, but not 70.) Julia automatically assigns a type when it parses a value. a `1` will be an integer, a `1.0` an floating point number. Rational numbers are made by using two division symbols, `1//2`.

For many operations the type will be conserved, such as adding to integers. For some operations, the type will be converted, such as dividing two integer values. Mathematically, we know we can divide some integers and still get an integer, but `julia` usually opts for the same output for its functions (and division is also a function) based on the type of the input, not the values of the input.

Okay, maybe that is too much. Let's get started.

Expressions

- Compute the following value:

$$(5/9)(-10-32)$$

Enter a number: ______________________________

- Compute the following value:

$$9/5(100)+32$$

Enter a number: ______________________________

- Compute the following value:

$$-4.9 \cdot 9^2 + 1.7 \cdot 9 + 8.6$$

Enter a number: ______________________________

- Compute the following value:

$$\frac{1 + 2 \cdot 3}{4 + 5^6}$$

Enter a number: ______________________________

Math functions

- Compute the following value:

$$\sqrt{0.61 \cdot (1-0.61)/100}$$

Enter a number: ______________________________

- Compute the following value (here math notation and computer notation are not the same):

$$\cos^2(\pi/3)$$

Enter a number: ____________________

- Compute the following value:

$$\sin^2(\pi/3) \cdot \cos((\pi/6)^2)$$

Enter a number: ____________________

- Compute the following value:

$$e^{(1/2)\cdot(3-1.8)^2}$$

Enter a number: ____________________

- Compute the following value:

$$1 + \frac{1}{2} + \frac{1}{2\cdot 3} + \frac{1}{2\cdot 3\cdot 4} + \frac{1}{2\cdot 3\cdot 4\cdot 5}$$

Enter a number: ____________________

- Compute the following value (`cosd` takes degree arguments, `cos` takes radian values):

$$\frac{5}{\cos(54^\circ)} + \frac{8}{\sin(54^\circ)}$$

Enter a number: ____________________

- In mathematics a function is defined not only by a rule but also by a *domain* of possible values. Similarly with `julia`. What kind of error does `julia` respond with if you try this command: `sqrt(-1)`?

Your answer:

Precedence

- There are 5 operations in the following expression. Write a similar expression using 4 pairs of parentheses that evaluates to the same value:

$$1 - 2 + 3 \cdot 4^5/6$$

Your answer:

- Which of these will also produce $1/(3 \cdot 4)$:

Make a selection:

1. `1 / 3 * 4`
2. `1 / 3 / 4`
3. `1 * 3 / 4`

Variable

- Let `x=5` and `y=8` compute

$$x - \sin(x + y)/\cos(x - y)$$

Enter a number: ______________________________

- For the polynomial

$$y = ax^2 + bx + c$$

Let $a = 0.00014$, $b = 0.61$, $c = 649$, and $x = 200$. What is y?

Enter a number: ______________________________

- If

$$\frac{\sin(\theta_1)}{v_1} = \frac{\sin(\theta_2)}{v_2}$$

and $\theta_1 = \pi/5$, $\theta_2 = \pi/6$, and $v_1 = 2$, find v_2.

Enter a number: ____________________________

Some applications

- The period of simple pendulum depends on a gravitational constant $g = 9.8$ and the pendulum length, L, in meters, according to the formula: $T = 2\pi\sqrt{L/g}$.

A rope swing is timed to have a period of 6 seconds. How long is the length of the rope if the formula applies?

Enter a number: ____________________________

- An object dropped from a building of height h (in feet) will fall according to the laws of projectile motion:

$$y(t) = h - 16t^2$$

If $h = 51$ find y if $t = 1.5$.

Enter a number: ____________________________

- Suppose $v = 2 \cdot 10^8$ and $c = 3 \cdot 10^8$ compute

$$\frac{1}{\sqrt{1 - v^2/c^2}}$$

(Be careful, this expression from a theory of relativity is susceptible to *integer* overflow on some computers!)

Enter a number: ____________________________

Trig practice

- A triangle has sides $a = 500$, $b = 750$ and $c = 901$. Is this a right triangle?

Make a selection:

1. Yes

2. No

- The law of sines states for a triangle with angle A, B, and C and opposite sides labeled a, b, c one has

$$\sin(A)/a = \sin(B)/b = \sin(C)/c.$$

If $A = 115°$, $a = 123$, and $b = 16$, find B (in degrees).

Enter a number: ____________________________

- The law of cosines generalizes Pythagorean's theorem: $c^2 = a^2 + b^2 - 2ab\cos(C)$. A triangle has sides $a = 5$, $b = 9$, and $c = 8$. Find the angle C (in radians)

Enter a number: ____________________________

Numbers

Scientific notation represents real numbers as $a \cdot 10^b$, where b is an integer, and a may be a real number in the range -1 to 1. In `julia` such numbers are represented with an `e` to replace the 10, as with `1.2e3` which would be $1.2 \cdot 10^3$ (1,230) or `3.2e-1`, which would be $3.2 \cdot 10^{-1}$ (0.32).

- The output of `sin(pi)` in `julia` gives `1.2246467991473532e-16`. Is this number

Make a selection:

1. close to -1.22
2. close to 0
3. close to 1.22

- Which number is larger? `9e-10` or `7e8`?

Make a selection:

1. `7e8`
2. `9e-10`

- Is `7e-10` greater than `8e-9`?

Make a selection:

1. Yes
2. No

- Which number is closest to `1.23e-4`?

Make a selection:

1. 1/10
2. 1000
3. 1/1000
4. 1/10000

- What is the sum of `12e3` and `32e-1`?

Enter a number: ____________________________

- The value `5e-1` is just:

$$2^{-1}.$$

Compute the value using `^`.

(This isn't quite as easy as it looks, as the output of the power function (`^`) depends on the type of the input variable.)

What command did you use:

Your answer:

Julia has different storage type for integers (which are stored exactly, but have smaller bounds on their size); rational numbers (which are stored exactly in terms of a numerator and a denominator); real numbers (which are *approximated* by floating point numbers); and complex numbers (which may have either have integer or floating point values for the two components.) When `julia` parses a value, it will determine the type by how it is entered.

- For example, the values `2`, `2.0`, `2 + 0im` and `2//1` are all the same and yet all different. What type is each?

Your answer:

CHAPTER 2

Functions in Julia

Read more about this here. We begin by loading the `MTH229` package:

```
using MTH229
```

For the impatient:

A *function* in mathematics is defined as "a relation between a set of inputs and a set of permissible outputs with the property that each input is related to exactly one output." That is a general definition. Specialized to mathematical functions of one real variable returning a real value, we can define a function in terms of a rule, such as:

$$f(x) = x^2 - 2.$$

The **domain** is the set of all permissible values for x, in this case all x, but this need not be the case either due to the rule not being defined for some x or a more explicit restriction, such as $x \geq 0$. The **range** is the set of all possible outputs. Written in set notation, this is $\{f(x) : x \in \text{ the domain }\}$.

Mathematically, we evaluate or call a function with the notation $f(2)$ or $f(3)$, say.

Mathematically we might refer to the function by its name, f, or its values $f(2)$, ...

In `Julia` basic mathematical functions are defined and used with the *exact* same notation. This creates a function `f`:

```
f(x) = x^2 - 2
```

```
f (generic function with 1 method)
```

Unlike an expression, the value `x` in is not needed to be defined until we call the function. As with math, this variable name need not be `x` it could be `y` or `theta` though it often is.

We can call `f` for the value of `2` with:

```
f(2)
```

```
2
```

That is, as with typical mathematical notation, the function is "called" by passing a value to it with parentheses.

Within a cell, we can evaluate one or more values by using commas to separate them:

```
f(1), f(2), f(3)
```

```
(-1,2,7)
```

The function name refers to the function object:

```
f
```

```
f (generic function with 1 method)
```

Don't worry about the words "generic" and "method", but be aware that because of this you can't rename a function into a variable, without an error. As well, `Julia` isn't even very keen on reusing a function name for another function and may give a warning.

Functions can be more complicated than the "one-liners" illustrated. In that case, a multiline form is available:

```
function fn_name(args...)
  body
end
```

The keyword `function` indicates this is a function whose name is given in the definition. Within the body, the last expression evaluated is the ouput, unless a `return` statement is used.

For basic uses of functions 90

However, there are some finer details that do arise from time to time, as explained later on.

Questions

- Define the following function:

```
f(x) = exp(-x) * sin(x)
```

```
f (generic function with 1 method)
```

Find the values $f(1)$ and $f(e)$:

The value of $f(1)$ is:

Enter a number: ______________________________

The value of $f(e)$ is:

Enter a number: ______________________________

- Define the function

```
f(x) = 5/sin(x) + 8/cos(x)
```

```
f (generic function with 1 method)
```

Which value is greater? $f(\pi/6)$ or $f(\pi/3)$?

Make a selection:

1. $f(\pi/3)$
2. $f(\pi/6)$

- Write a function that describes a line with slope 1.5 going through the point $(3, 1)$. What is the value of $f(10)$?

The function is:

Your answer:

The value of $f(10)$ is:

Enter a number: ______________________________

- Write a function to convert Celsius to Fahrenheit $F = 9/5C+32$. Use it to find the Fahrenheit value when $C = 56.7$ and when $C = -89.2$. (The record high and low temperatures.)

The function is

Your answer:

The value at $C = 56.7$ is

Enter a number: ____________________________

The value at $C = -89.2$ is:

Enter a number: ____________________________

- Write a function that computes

$$f(x) = 10x^2 - 3x - 7 - \frac{1}{x}$$

Use it to find the values of $f(1)$, and $f(3)$.
The function is defined by:

Your answer:

The value $f(1)$ is

Enter a number: ____________________________

The value $f(3)$ is

Enter a number: ____________________________

- Write a function that computes:

$$f(t) = A\sin(Bt - C) + D$$

where $A = 3.1$, $B = 2\pi/365$, $C = 1.35$, and $D = 12.12$.
This function models the amount of daylight in Boston when t records the day of the year. How much daylight is there for $t = 1$, $t = 365/2$, $t = 35$?
The function is

Your answer:

The value at $t = 1$ is

Enter a number: ______________________________

The value at $t = 365/2$ is:

Enter a number: ______________________________

The value at $t = 35$ is:

Enter a number: ______________________________

- Person A starts at the origin and moves west at 60 MPH. Person B starts 200 miles north of the origin and moves south at 70 MPH. Write a function that computes the distance between the two people as a function of t in minutes.

(The (x, y) position of person A is $(60 \cdot t/60), 0)$ and the (x, y) position of person B is $(0, 200 - 70 \cdot t/60)$. Use the distance formula to write a function.)
The distance at $t = 0$ is:

Enter a number: ______________________________

The distance at $t = 30$ is:

Enter a number: ______________________________

The distance at $t = 120$ is:

Enter a number: ______________________________

- A specific "Norman" window is a square window with a half circle on top. If the length of the side of the square is x, write a function describing the total area:

Your answer:

For such a Norman window with side $x = 2$, what is the area?

Enter a number: ______________________________

Cases

Some functions are defined in terms of cases. For example, a cell phone plan might depend on the data used through:

> The amount is 40 dollars for the first 1 Gb of data, and 10 dollars more for each *additional* Gb of data.

This function has two cases to consider: one if the data is less than 1 Gb and the other when it is more.

How to write this in `julia`?

The ternary operator `predicate ? expression1 : expression2` has three pieces: a predicate question, such as `x < 10` and two expressions, the first is evaluated if the predicate is `true` and the second if the predicate is `false`. They are useful to write functions that are defined by cases. (They are a light-weight form of the traditional `if-then-else` construct.)

The example above, could then be done with:

```
f(d) = d <= 1.0 ? 40.00 : 40.00 + 10.00 * (d - 1.0)
```

```
f (generic function with 1 method)
```

- Use the ternary operator to write a function $f(x)$ which takes a value of x when x is less than 10 and is otherwise a constant value of 10.

Your answer:

- A function is given by the rule: it is x^2 if $x > 1$ and otherwise, x.

Express this in Julia using the ternary operator.

Your answer:

- Write a function to express the following: If a person buys up to 100 units the cost per unit is 5 dollars, for every additional unit beyond 100 the cost is 4 dollars. The function should return the total cost to buy x units. (Use the ternary operator with `x <= 100` as the condition.)

Your answer:

Composition

Composition of functions is a useful means to break complicated problems into easier to solve ones. The math notation is typically $f(g(x))$ and in `julia` this is no different. When thinking about the operation of composition, the notation $f \circ g$ is used. For that, there isn't any built-in `julia` notation.

- For the function $h(x) = ((x+1)/(x-1))^{2/3}$ write this as the composition of two functions $f(x)$ and $g(x)$. Use these to evaluate $h(3)$. Show your work and the answer.

  Your answer:

- Write the function $h(x) = (\cos(12x))^3$ as the composition of two functions $f(x)$ and $g(x)$ and use these to evaluate $h(2)$. Show your work and the answer.

  Your answer:

Parameters: Functions can have keyword arguments.

In the following, we define a function which evaluates a linear equation. A familiar description of a line has two parameters: the slope m and the y intercept, b: $y = mx + b$. The main focus is on x, but the values of m and b give each problem a different application. To keep these separate, keyword arguments may be given. In the following, `m` and `b` are passed in via keywords and, as defined, have default values of `m=1` and `b=0`.

```
mxplusb(x; m=1, b=0) = m*x + b
mxplusb(10, m=3, b=4)              # arguments are named
```

```
34
```

The semicolon separates the two types of arguments.

If no value are passed, the defaults are used:

```
mxplusb(10, m=2)   # evaluates 2 * 10 + 0
```

```
20
```

- The formula for a catenary has a parameter a:

$$y = a \cosh(x/a)$$

($\cosh(x)$ is the hyperbolic cosine, defined by $(1/2) \cdot (e^x + e^{-x})$ and implemented by `cosh`.)

Write a function, `c(x;a=1)`, with `a` as a parameter defaulting to 1. Compute `c(1)`, `c(1,a=2)`, and `c(1, a=1/2)`.

The function is:

Your answer:

The value of `c(1)`:

Enter a number: ____________________

The value of `c(1, a=2)`:

Enter a number: ____________________

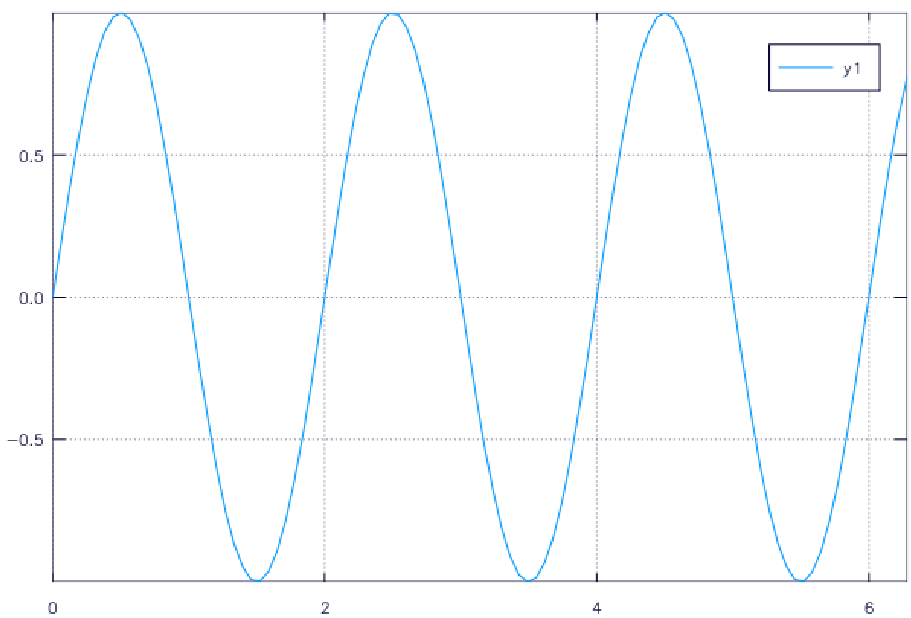
The value of `c(1, a=1/2)`:

Enter a number: ____________________

Functions can be used as arguments to other functions:

Like many computer languages, `Julia` allows functions to be arguments to functions. In the following, we see we can plot a function by passing in a function object to a `plot` function. Notice, the variable name (`f`) and not the call (`f(x)`) is given to the `plot` command (from the `Plots` package, loaded with `MTH229`):

```
f(x) = sin(pi * x)
plot(f, 0, 2pi)                    # pass function for first argument plot(f,a,b)
```

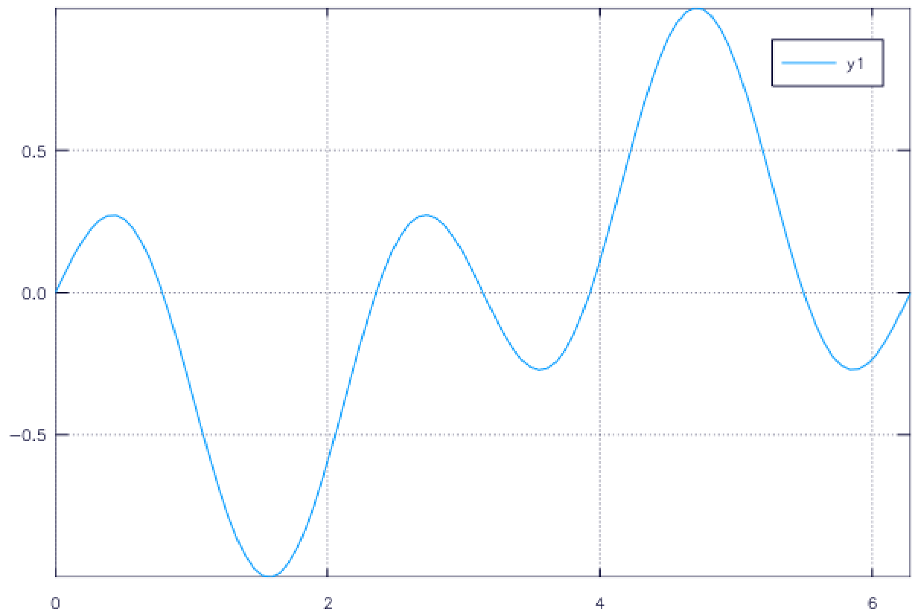


Using functions in this way will form a *very* common pattern.

Functions can be anonymous.

When functions are passed into other functions, it can be convenient to define them without names. The syntax is slightly different, but follows mathematical notation: `x -> expression_in_x`, or more generally `args -> body`. Such functions are called *anonymous functions*:

```
plot(x -> sin(x)*cos(2x), 0, 2pi)
```



Returning a function

Familiar mathematical functions take a real number as input and return a real number. However, the concept of a function is more general. With `julia` it is useful to write functions that take functions as arguments, and return a derived function as an output. For the return value an *anonymous function* is typically used.

- Describe what the following function does to the argument f, when f is a function. (There isn't anything to do but recognize that `n` takes a function as input and returns a function as output, this question is how is `n(f)` related to `f`.)

```
n(f::Function) = x -> -f(x)
```

```
n (generic function with 1 method)
```

Your answer:

- This function takes a function and two points and returns a new function that evaluates the secant line:

```
function secant(f, a, b)
  m = (f(b) - f(a)) / (b - a)
  x -> f(a) + m * (x - a)
end
```

```
secant (generic function with 1 method)
```

Let $f(x) = \sin(x)$. Let $a = \pi/6$ and $b = \pi/3$. Show that the secant line at $\pi/4$ is less than the function value at $\pi/4$ by computing both:

The function at $\pi/4$ is:

Enter a number: ____________________________

The secant line's value at $\pi/4$ is:

Enter a number: ____________________________

Some other facts about functions

Julia functions can have more than one variable

For example, this function is used to compute the area of a rectangle:

```
rectarea(b, h) = b * h                # area of rectangle is base times height
```

```
rectarea (generic function with 1 method)
```

The `log` function is an example, where `log(b,x)` will find the log base b of x, while `log(x)` uses the default base `e`, or the natural log.

Julia functions with different signatures, can have the same name!

This finds the area of a square using the previously defined rectangle function:

```
rectarea(b) = rectarea(b,b)         # area of square using area of rectangle
```

```
rectarea (generic function with 2 methods)
```

Which function is used depends on the arguments that are passed when calling the function. Hence, `log(x)` will find log base e of x whereas `log(10, x)` can be used to find log base 10.

CHAPTER 3

Graphics with Julia

Read about this here: Graphing Functions with Julia.

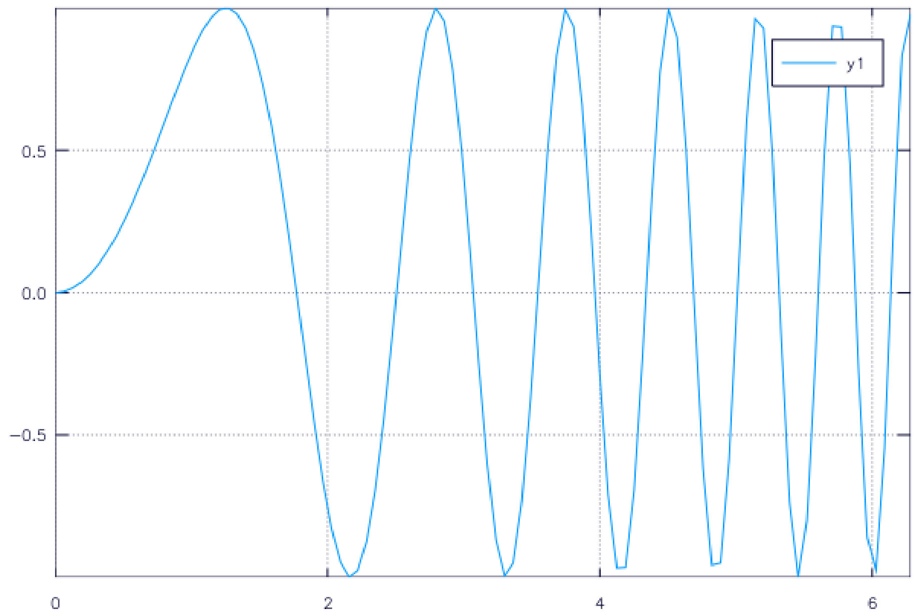
For the impatient, `julia` has several packages that allow for graphical presentations, but nothing "built-in." We will use the `Plots` package, a front-end to several graphing packages. As a backend we have to choose one, and will use `plotly`. (We assume this is the default, if not, enter the command `plotly()` to select it.)

The `Plots` package is loaded when the `MTH229` package is:

```
using MTH229
```

The Plots package brings in a `plot` function that makes plotting functions as easy as specifying a function object and the x domain to plot over:

```
f(x) = sin(x^2)
plot(f, 0, 2pi)                    # plot(f, a, b)
```

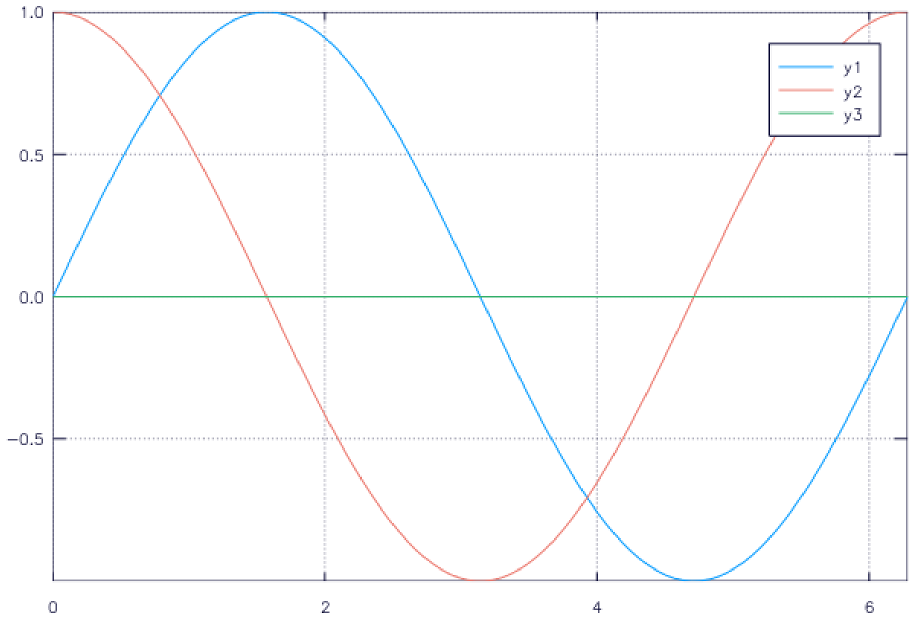


Often most of the battle is *judiciously* choosing the values of a and b so that the graph highlights a feature of interest. Such as a relative maximum or minimum, a zero, a vertical asymptote, a horizontal asymptote, a slant asymptote...

The use of a function as an argument is not something done with a calculator, but is very useful when using `julia` for calculus as many actions may be viewed as operating on the function f, not the values of the function, $f(x)$.

More than one function can be plotted on a graph. The `plot!` function makes this easy: make the first plot with `plot` and any additional ones with `plot!`. For example:

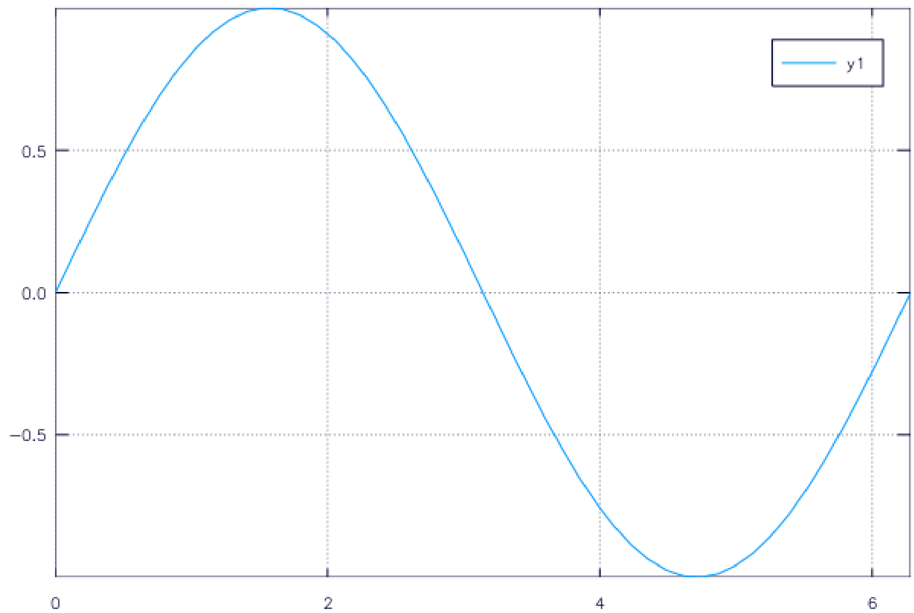
```
plot(sin, 0, 2pi)
plot!(cos, 0, 2pi)
plot!(zero, 0, 2pi)           # add the line f(x) = 0.
```



(You can even just call `plot!(cos)` or `plot!(zero)` and implicitly get the x-range from the current graph.)

A plot is nothing more than a connect-the-dot graph of paired x and y values. It can be useful to know how to do the steps. The above graph of `sin` could be done with:

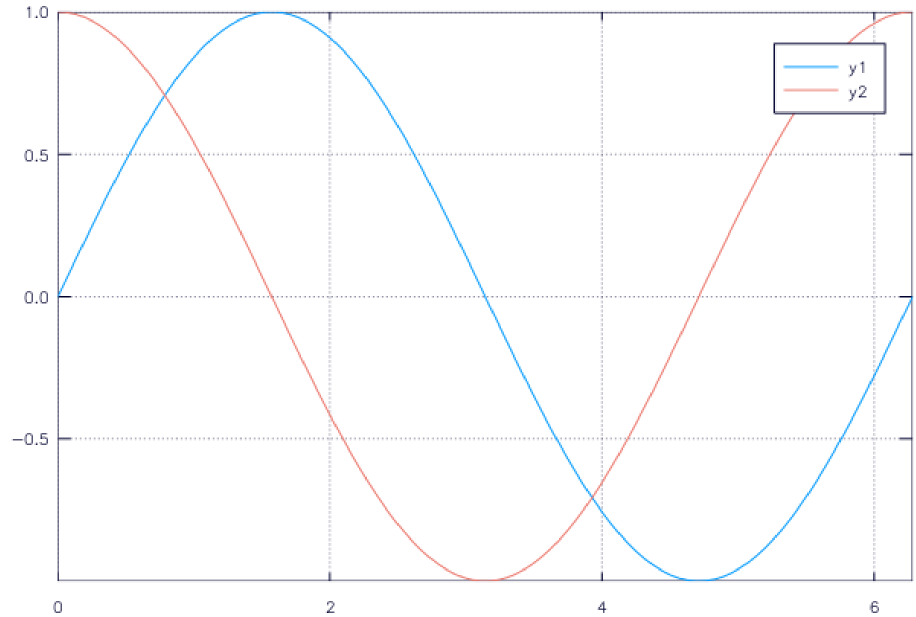
```
a, b = 0, 2pi
xs = linspace(a, b)                     # 50 points between a and b
ys = map(sin, xs)                   # or ys = [sin(x) for x in xs] or sin.(xs) (see the notes
plot(xs, ys)
```



The xs and ys are written as though they are "plural" because these variables contain 50 values each in a container (a vector in this case). The map command "maps" a function to each value in the container and returns a new container with the "mapped" values. In the example above, these are the values for the sin at each x.

Containers (vectors in this case) are often constructed by combining like values within square brackets separated by commas: e.g., [a,b]. For plotting, we can combine functions using [] and all will plot, as an alternative to using plot!:

```
plot([sin, cos], 0, 2pi)
```



Finally, `scatter!` can be used to add points to a graph. These are specified as vectors of `x` and `y` values.

Questions

- Make a plot of $f(x) = \exp(x) - x^3$ over the interval $[3, 5]$. From your graph, estimate the value where the graph crosses the x axis.

The commands to produce the plot are:

Your answer:

The approximate zero is:

Enter a number: ______________________________

- For the same function $f(x) = \exp(x) - x^3$ make graphs over different domains until you can find another zero. What is this other approximate zero?

Enter a number: ______________________________

- Graph the polynomial function $f(x) = 2x^3 - 5x^2 + x$. By graphing different domains, approximate the location of the three roots to one decimal point.

  The smallest root is:

Enter a number: ______________________________

  The middle root is:

Enter a number: ______________________________

  The largest root is:

Enter a number: ______________________________

- A cell phone plan has 700 minutes of talking for 20 dollars with each additional minute over 700 minutes costing 10 cents per minute. Write a function representing this rate for any positive time t. Then graph the function between 0 and 1000.

  Your answer:

- The function $f(x) = (\sin(x)^2 - 2x + 1)^5$ is very flat between -1 and 2. By repeatedly graphing on smaller intervals, find an interval of the type $[x, x + 0.01]$ which contains a zero. (E.g., $[0.68, 0.69]$.)

Your answer:

- The function $f(x) = (2x - x^2) \cdot e^x$ increases on just one interval. What is it? (Use interval notation (a, b).)

Your answer:

- The function $f(x) = \sin(120\pi x)$ is a highly oscillatory function. Using trial-and-error, or some other means, find a value b so that the plot over $[-b, b]$ shows exactly one period.

Enter a number: ______________________________

Asymptotes

Function with asymptotes (vertical, horizontal, or slant) can pose challenges, as the the wrong choice of domain to plot over can mean the plotting of points on a vertical asymptote can overwhelm other values. Judiciously choosing the values to plot over is important. For example, plotting the following function over $[0, 2]$ will show the vertical asymptote (spuriously plotted), but not whether there is a slant or horizontal asymptote. For that, try plotting over $[-10, 10]$.

- Graph the rational function $f(x) = (x^2 + 1)/(x - 1)$. Do you see any asymptotes (horizontal, slant, vertical)? If so, describe them.

  Your answer:

- Make a graph of the rational function $f(x) = (x^2 - 2x + 1)/(x^2 - 4)$. Use a suitable domain so that any horizontal asymptotes can be seen. What commands did you use?

Your answer:

- Make a plot of $f(x) = \tan(x)$ over $(-\pi/2, \pi/2)$. From your graph, what x value corresponds to a y value of 1.1? (Plotting with `a=-pi/2, b=pi/2` will give an unpleasant graph. Try backing off a bit from each side.)

Enter a number: ____________________

- Make a plot of $f(x) = \cos(x)$ and $g(x) = 1 - x^2/2$ over $[-\pi/2, \pi/2]$. How many times to the graphs intersect? Can you even tell? If not, why not?

Your answer:

- (Transformations of graphs) On the same graph, plot both `f(x) = max(0, 1-abs(x))` and `g(x) = 1 + 2*f(x-3)`. Describe the relationship of `g` and `f` in terms of the values 1, 2 and 3. (shift up, down, scale, ...)

Make a selection:

1. The shape of the graph of g is the same as the shape of the graph of f, but shifted up 1, right 3 and stretched by 2
2. The shape of the graph of g is the same as the shape of the graph of f, but shifted up 3, right 1 and stretched by 2
3. The shape of the graph of g is the same as the shape of the graph of f, but shifted up 1, right 2 and stretched by 3
4. The shape of the graph of g is the same as the shape of the graph of f, but shifted up 2, right 1 and stretched by 3

NaN values.

The value `NaN` is a floating point value that arises during some indeterminate operations, such as `0/0`. The `plot` function will stop connecting the dots when it encounters an `NaN` value. This can be useful. The following uses it to graph a straight line *only* when the cosine is positive.

- Make a plot of `f(x) = sin(x)` and `g(x) = cos(x) > 0 ? 0.0 : NaN` over $[0, 2\pi]$. What is the relationship? (Notice, the graph of $g(x)$ shows only when $\cos(x)$ is positive.)

Your answer:

- The following function can be used to restrict the range of a mathematical function:

```
trim(f::Function; cutoff=10) = x -> abs(f(x)) > cutoff ? NaN : f(x)
```

```
trim (generic function with 1 method)
```

Try plotting `trim(f)` when $f(x) = (x^2 - 2x + 1)/(x^2 - 4)$ over $[-5, 5]$. What do you see as compared to the previous graph of the rational function $f(x)$?

Your answer:

Creating sequences

`Julia` has different ways to create sequences of numbers. One of them is `linspace`. The `linspace(a, b)` command creates (by default) 50 evenly spaced values between `a` and `b`. The `linspace` command provides a useful set of values to use when plotting using the lower-level commands. (See `?colon` for another.)

- write a simple command to produce 50 values between 0 and 2π

Your answer:

- Write a simple command to produce 50 evenly-spaced values between 1/10 and 10.

Your answer:

mapping a function

`Julia` has different ways of applying a function to each value in a collection. Some functions, like `sin` are vectorized to naturally do so, others are not. The `map` function, called as `map(f, collection)` will work. There are also list comprehensions and the newer "dot" notation.

- If `a = [1,2,3,4,5]` find `a^3` for each value. (Use the `map` function.)

Your answer:

- The command `xs = linspace(0, 10pi)` creates many points between 0 and `10pi`. Map the function $f(x) = \cos^2(x^{1/2})$ to these values. Write your commands here:

Your answer:

CHAPTER 4

Finding Zeros of Functions

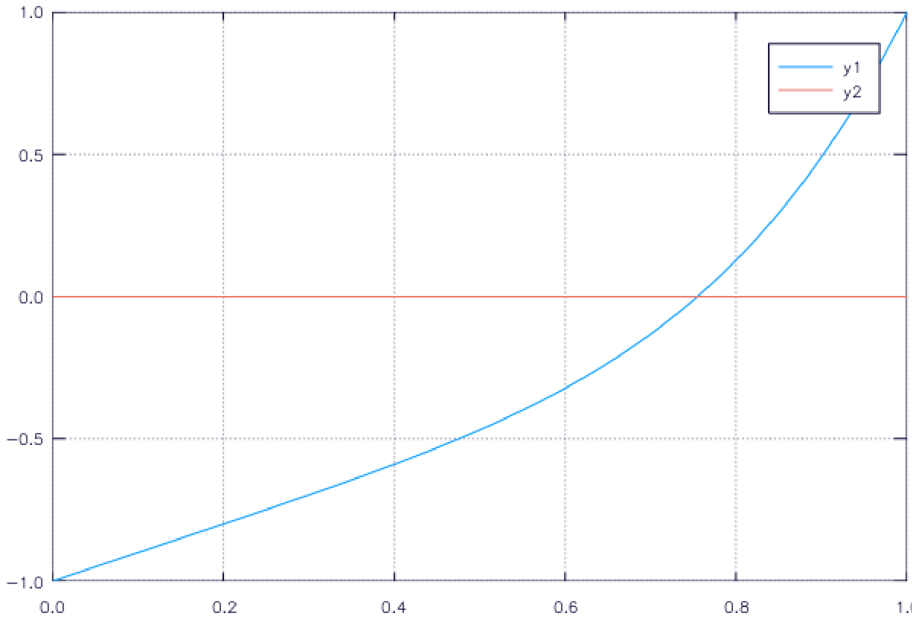
Read about this topic here: Solving for zeros with julia.

For the impatient, these questions are related to the zeros of a real-valued function. That is, a value x with $f(x) = 0$. The `Roots` package of `Julia` will provide some features. This is loaded when `MTH229` is:

```
using MTH229
```

Graphically, a zero of the function $f(x)$ occurs where the graph crosses the x-axis. Without much work, a zero can be *estimated* to a few decimal points. For example, we can zoom in on the zero of $f(x) = x^5 + x - 1$ by graphing over $[0, 1]$:

```
f(x) = x^5 + x - 1
plot(f, 0, 1)
plot!(zero, 0, 1)
```



We can see the answer is near 0.7. We could re-plot to get closer, but if more accurate answers are needed, numeric methods, such as what are discussed here, are preferred.

The notes talk about a special case - zeros of a polynomial function. Due to the special nature of polynomials, there are many facts known about the zeros. A typical example is the quadratic equation which finds both answers to any quadratic polynomial. These facts can be exploited to find roots. The `Roots` package provides the `roots` function to *numerically* find all the zeros of a polynomial function (real and complex) and the `fzeros` function to find just the real roots. (The heavy lifting here is done by the `Polynomials` package.)

```
f(x) = x^5 + x - 1
roots(f)      ## all roots
```

```
5-element Array{Complex{Float64},1}:
 -0.877439+0.744862im
 -0.877439-0.744862im
       0.5+0.866025im
       0.5-0.866025im
  0.754878+0.0im
```

```
fzeros(f)     ## real roots only
```

```
1-element Array{Real,1}:
 0.754878
```

(Notice that in both cases the argument is a function. This is a recurring pattern in these projects: A function is operated on by some action which is encapsulated in some function call like `roots`.)

Bisection method

For the general case, non-polynomial functions, the notes mention the bisection method for zero-finding. This is based on the *intermediate value theorem* which guarantees a zero for a continuous function $f(x)$ over any interval $[a, b]$ *when* $f(a)$ and $f(b)$ have *different* signs. Such an interval is called a **bracket** or bracketing interval.

The algorithm finds a zero by successive division of the interval. Either the midpoint is a zero, or one of the two sub intervals must be a bracket.

The `bisection_viz` function in the `MTH229` package provides an illustration.

The notes define a `bisection` method and a stripped down version is given below. More conveniently the `Roots` package implements this in its `fzero` function when it is called through `fzero(f, a, b)`. For example,

```
f(x) = x^2 - 2
fzero(f, 1,2)                           # find sqrt(2)
```

```
1.4142135623730951
```

As mentioned, for polynomial functions the `fzeros` function finds the real roots. In general, the `fzeros` function will try to locate real roots for any function but it needs to have an interval in which to search. For example this call will attempt to find all zeros within $[-5, 5]$ of $f(x)$:

```
f(x) = x^2 - 2
fzeros(f, -5, 5)
```

```
2-element Array{Real,1}:
 -1.41421
  1.41421
```

[This function will have issues with non-simple roots and with roots that are very close together, so should be used with care.]

This summary of functions in the `Roots` package might help:

- The call `roots(f)` finds all roots of a polynomial function, even complex ones.
- The call `fzeros(f)` finds all *real* roots of polynomial function.
- The call `fzero(f, a, b)` finds a root of a function between a **bracketing** interval, `[a,b]`, using the bisection method. This *method* is guaranteed to work if a bracket is given.
- The call `fzeros(f, a, b)` function *attempts* to find all roots of a function in an interval `[a,b]`. This may miss values; answers should be checked graphically.

Questions to answer

Polynomial functions

- Find a zero of the function $f(x) = 212 - 0.65x$.

Enter a number: ____________________________

- The parabola $f(x) = -16x^2 + 200x$ has one zero at $x = 0$. Graphically find the other one. What is the value

Enter a number: ____________________________

- Use the quadratic equation to find the roots of $f(x) = x^2 + x - 1$. Show your work.

  Your answer:

- Use the `roots` function to find the zeros of $p(x) = x^3 - 4x^2 - 7x + 10$. What are they?

Make a selection:

1. `-2.33333, 0.0`
2. `-2.0, 1.0, 5.0`
3. `-0.788376+1.08241im,  -0.788376-1.08241im, 5.57675+0.0im`

- Use the `fzeros` function to find the *real* zeros of $p(x) = x^5 - 5x^4 - 2x^3 + 13x^2 - 17x + 10$. (The `roots` function returns all 5 zeros guaranteed by the Fundamental Theorem of Algebra, not all of them are real.)

Make a selection:

1. `-2.0, 1.0, 5.0`
2. `0.0, 2.0, 6.0`
3. `-2.000 + 0.0im, 0.500 + 0.866im, 0.500 - 0.866im, 1.0 + 0.0im, 5.0 + 0.0im`
4. `1.0, 5.0`

- Descarte's rule of signs allows one to estimate the number of *positive* real roots of a real-valued polynomial simply by counting plus and minus signs. Write your polynomial with highest powers first and then count the number of changes of sign of the coefficients. The number of positive real roots is this number or this number minus $2k$ for some k.

Apply this rule to the polynomial $x^5 - 4x^4 + 5x^3 - 16x^2 - 3$. What is the maximal possible number of positive real roots? What is the exact number?

The maximal possible number of real roots is:

Enter a number: ____________________________

The actual number of positive real roots is:

Enter a number: ____________________________

Other types of functions

- Graph the function $f(x) = x^2 - 2^x$. Try to graphically estimate all the zeros. Answers to one decimal point.

Make a selection:

1. `2.0, 4.0`
2. `-1.414, 1.414`
3. `0.0, 2.0`
4. `-1.0, 1.0`

- Graphically find the point(s) of intersection of the graphs of $f(x) = 2.5 - 2e^{-x}$ and $g(x) = 1 + x^2$.

Your answer:

- The MTH229 package provides a bisection method, here is an abbreviated version:

```
function bisection(f, a, b)
  @assert f(a) * f(b) < 0        # an error if [a,b] is not a bracket

  mid = a + (b - a) / 2

  while a < mid < b
    if f(mid) == 0.0 break end
    f(a) * f(mid) < 0 ? (b = mid) : (a = mid)
    mid = a + (b - a) / 2
  end
  mid
end
```

```
bisection (generic function with 1 method)
```

The function above **starts** with two values, a and b with the property that $f(a)$ and $f(b)$ have different signs, hence if $f(x)$ is continuous, it must cross zero between a and b. The algorithm simply bisects this interval by finding `mid`. It then selects either `[a,mid]` or `[mid,b]` to be the new interval where a zero is guaranteed. It **stops** if the interval is too small to subdivide. This is an impossibility mathematically, but is the case with floating point numbers.

The `bisection` function is used to find a zero, when $[a, b]$ brackets a zero for f. It is called like `bisection(f, a,b)`, for suitable `f`, `a`, and `b`.

- Let $f(x) = \sin(x)$. The interval $[3, 4]$ is a bracketing interval. What would the interval be after one step of the bisection method?

Make a selection:

1. $[3, 31/2]$

2. $[31/2, 4]$

What would the interval be after *three* steps:

Make a selection:

1. $[3, 31/8]$
2. $[31/8, 31/4]$
3. $[31/4, 33/8]$
4. $[33/8, 31/2]$
5. $[31/2, 35/8]$
6. $[35/8, 33/4]$
7. $[33/4, 37/8]$
8. $[37/8], 4]$

- Use the `bisection` function to find a zero of $f(x) = \sin(x)$ on $[3, 4]$. Show your commands and both the zero (`x`) and its value `f(x)`.

Your answer:

- Let $f(x) = \exp(x) - x^5$. In the long run the exponential dominates the polynomial and this function grows unbounded. By graphing over the interval $[0, 15]$ you can see that the largest zero is less than 15. Find a bracket and then use `bisection` to identify the value of the largest zero. Show your commands.

Enter a number: ______________________________

- Find the intersection point of $4 - e^{x/10} = e^{x/15}$ by first graphing to see approximately where the answer is. From the graph, identify a bracket and then use `bisection` to numerically estimate the intersection point.

Enter a number: ____________________________

- The bisection algorithm can't distinguish a vertical asymptote from a zero! What is the output of trying the bisection algorithm on $f(x) = 1/x$ over the bracketing interval $[-1, 1]$?

Enter a number: ____________________________

What is the reason for this:

Make a selection:

1. The intermediate value theorem does not apply as $[-1, 1]$ is not a bracketing interval
2. The intermediate value theorem does not apply as $f(x)$ is not continuous on $[-1, 1]$
3. This is not a good example, as the bisection does find an answer of zero

- The `Roots` package has a built-in function `fzero` that does different things, with one of them being a (faster) replacement for the `bisection` function. That is, if `f` is a continuous function and `[a,b]` a bracketing interval, then `fzero(f, a, b)` will do the bisection method until a zero is found or the interval can no longer be subdivided.

Show that `fzero(f,a,b)` works by finding a zero of the function `f(x) = (1 + (1 - n)^2)*x - (1 - n*x)^2` when $n = 8$. Use $[0, 0.5]$ as a bracketing interval. What is the value?

Enter a number: ____________________________

- The `airy` function is a special function of historical importance. (It is a built-in function.) Find its largest *negative* zero by first plotting, then finding a bracketing interval and finally using `fzero` to get a numeric value.

From your graph, what is a suitable *bracketing* interval?

Make a selection:

1. $[-10, 10]$
2. $[-2, 2]$
3. $[0, 2\pi]$

4. $[-5, 5]$
5. $[-3, 0]$

The value of the largest *negative* zero is:

Enter a number: ____________________________

- Suppose a crisis manager models the number of cases of water left after x days by $f(x) = 550,000 \cdot (1 - 0.25)^x$. When does the supply of water dip below $100,000$? Find a bracket and then use a numeric method to get a precise answer.

Your answer:

The `fzeros(f, a, b)` function basically splits the interval $[a, b]$ into lots of subintervals and then applies the bisection method to each to find the zeros of `f` over the interval. (Note the plural, this is not just `fzero`.) The algorithm is a bit crude, so may miss some zeros. Plotting is suggested to confirm the answers.

- Use `fzeros` to find all the zeros of $\cos(x) - 1/2$ over $[0, 4\pi]$.

Make a selection:

1. `-1.0472, 1.0472`
2. `1.5708, 4.71239, 7.85398, 10.9956`
3. `1.0472, 5.23599, 7.33038, 11.5192`
4. `1.0472, 5.23599`

- Use `fzeros` to find all the roots of $e^x = x^6$ over $[-20, 20]$.

Make a selection:

1. `-0.8155, 1.42961, 8.6131`
2. `1.85718, 4.53640`
3. `1.29586, 12.7132`
4. `-0.86565, 1.22689, 16.9989`

Using answers

The output of `fzeros` is a collection of values. It may be desirable to pass these onto another function. This is essentially *composition*. For example, we can check our work using this pattern:

```
f(x) = cos(x) + cos(2x)
zs = fzeros(f, [0,2pi])
map(f, zs)
```

```
2-element Array{Float64,1}:
 -3.33067e-16
  2.77556e-16
```

We see that the values are all basically 0, save for round-off error.

- Let $f(x) = 5x^4 - 6x^2$ and $g(x) = 20x^3 - 12x$. What are the values of f at the zeros of g? Show your commands and answer.

Your answer:

Issues with numerics

The `fzero` function implementing the bisection method is guaranteed to return a value where the function being evaluated crosses zero, though that value may not be an exact zero. However, this need not be the actual value being sought. This can happen when the function being evaluated is close to zero near the zero. For example, the function $(x-1)^5 = x^5 - 5x^4 + 10x^3 - 10x^2 + 5x - 1$ will be very flat near the one real zero, 1. If we try to find this zero with the expanded polynomial, we only get close:

```
f(x) = x^5 - 5x^4 + 10x^3 - 10x^2 + 5x - 1
fzero(f, .9, 1.1)
```

```
0.9994628906249997
```

- What zero does `fzero` return if instead of the expanded polynomial, the factored form `f(x) = (x-1)^5` were used?

Enter a number: ____________________________

- Make a plot of the expanded polynomial over the interval $[0.999, 1.001]$. How many zeros does the graph show?

Make a selection:

1. Just one at $x = 1$
2. Many zeros

- There can be numeric issues when roots of a polynomial are close to each other. For example, consider the polynomial parameterized by `delta`:

```
delta = 0.01
f(x) = (x-1-delta)*(x-1)*(x-1+delta)
fzeros(f)
```

```
3-element Array{Real,1}:
 1.0
 1.0
 1.01
```

The above easily finds three roots separated by `delta`. What happens if `delta` is smaller, say `delta = 0.0001`, so the three mathematical roots are even closer together? Are all three roots still found by `fzeros`?

Make a selection:

1. Yes
2. No

CHAPTER 5

Limits of Functions

To get started, we load the `MTH229` package so that we can make plots and use some symbolic math:

```
using MTH229
```

Quick background

Read about this material here: Investigating limits with Julia.

For the impatient, the expression

$$\lim_{x \to c} f(x) = L$$

says that the limit as x goes to c of f is L. If $f(x)$ is *continuous* at $x = c$, the $L = f(c)$. This is almost always the case for a randomly chosen c - but almost never the case for a textbook choice of c. Invariably with text books - though not always - we will have `f(c) = NaN` indicating the function is indeterminate at `c`. For such cases we need to do more work to identify if any such L exists and when it does, what its value is.

We can investigate limits three ways: analytically, with a table of numbers, or graphically. Here we focus on two ways: graphically or numerically.

Investigating a limit numerically requires us to operationalize the idea of x getting close to c and $f(x)$ getting close to L. Here we do this manually:

```
f(x) = sin(x)/x
f(0.1), f(0.01), f(0.001), f(0.0001), f(0.00001), f(0.000001)
```

```
(0.9983341664682815,0.9999833334166665,0.9999998333333416,0.9999999983333334,0.9999999999833332,0.999
```

From this we see a *right* limit at 0 appears to be 1.

We can put into a column, but wrapping things in braces:

```
[f(0.1), f(0.01), f(0.001), f(0.0001), f(0.00001), f(0.000001)]
```

```
6-element Array{Float64,1}:
 0.998334
 0.999983
 1.0
 1.0
 1.0
 1.0
```

The compact printing makes it clear, the limit here should be $L = 1$.

Limits when $c \neq 0$ are similar, but require points getting close to c. For example,

$$\lim_{x \to \pi/2} \frac{1 - \sin(x)}{(\pi/2 - x)^2}$$

has a limit of $1/2$. We can investigate with:

```
c = pi/2
f(x) = (1 - sin(x))/(pi/2 - x)^2
[f(c+.1), f(c+.001), f(c+.00001), f(c+.0000001), f(c+.000000001)]
```

```
5-element Array{Float64,1}:
 0.499583
 0.5
 0.5
 0.4996
 0.0
```

Wait, is the limit $1/2$ or 0? At first $1/2$ seems like the answer, but the last number is 0.

Here we see a limitation of tables - when numbers get too small, that fact that they are represented in floating point becomes important. In this case, for numbers too close to $\pi/2$ the value on the computer for `sin(x)` is just 1 and not a number near 1. Hence the denominator becomes 0, and so then the expression. (Near 1, the floating point values are about 10^{-16} apart, so when two numbers are within 10^{-16} of each other, they can be rounded to the same number.) So watch out when seeing what the values of $f(x)$ get close to. Here it is clear that the limit is heading towards 0.5 until we get too close.

For convenience, this function from the `MTH229` package can make the above computations easier to do:

```
function lim(f::Function, c::Real; n::Int=6, dir="+")
         hs = [(1/10)^i for i in 1:n] # close to 0
         if dir == "+"
           xs = c + hs
         else
           xs = c - hs
```

```
        end
        ys = map(f, xs)
        [xs ys]
end
```

```
lim (generic function with 1 method)
```

It use follows the common pattern: `action(function, arguments...)`. E.g.,

```
f(x) = (1 - sin(x))/(pi/2 - x)^2
lim(f, pi/2)
```

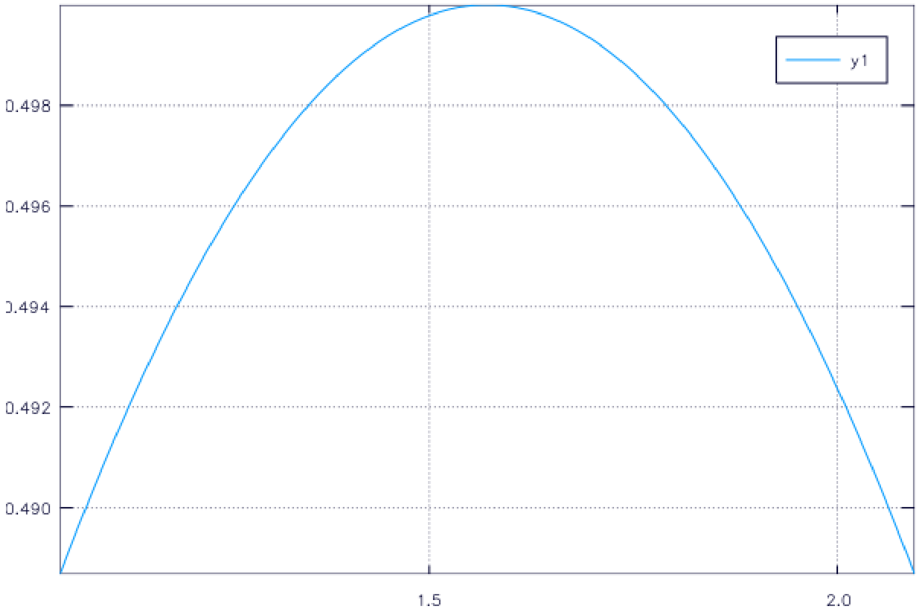
```
6x2 Array{Float64,2}:
 1.6708   0.499583
 1.5808   0.499996
 1.5718   0.5
 1.5709   0.5
 1.57081  0.5
 1.5708   0.500044
```

Graphical approach

The graphical approach is to plot the expression near c and look visually what $f(x)$ goes to as x gets close to c.

A graphical approach doesn't give as many significant digits, but won't have this floating point error. Here is a graph to investigate the same problem. We simply graph near c and look:

```
plot(f, c - pi/6, c + pi/6)
```



From the graph, we see clearly that as x is close to $\pi/2$, $f(x)$ is close to $1/2$. (The fact that `f(pi/2) = NaN` will either not come up, as `pi/2` is not among the points sampled or the `NaN` values will not be plotted.)

Questions: Graphical approach

- Plot the function to estimate the limit. What is the value?

$$\lim_{\theta \to 0} \frac{\sin(5\theta)}{\sin(2\theta)}.$$

Enter a number: ______________________

- Plot a function to estimate the limit. What is the value?

$$\lim_{x \to 0} \frac{2^x - \cos(x)}{x}.$$

Enter a number: ______________________

- Plot the function to estimate the limit. What is the value?

$$\lim_{\theta \to 0} \frac{\sin^2(4\theta)}{\cos(\theta) - 1}.$$

Enter a number: ______________________________

Questions: Tables

- This expression is indeterminate at 0 of the type 0/0:

$$\frac{1 - \cos(x)}{x}.$$

  What value does `julia` return if you try to evaluate it a 0?

Your answer:

- This expression is indeterminate at 0 of the type $0 \cdot \infty$:

$$x \log(x).$$

Your answer:

- This expression is indeterminate at 0 of the type 0^0:

$$x^{1/\log(x)}.$$

  What value does `julia` return?

Your answer:

- This expression is indeterminate at at $\pi/2$ of the type 0/0.

$$\frac{cos(x)}{\pi/2 - x}$$

  What value does `julia` return?

Your answer:

- Find the limit using a table. Show your commands.

$$\lim_{x \to 0+} \frac{\cos(x) - 1}{x}.$$

Your answer:

What is the estimated value of the limit?

Enter a number: ______________________________

- Find the limit using a table. What is the estimated value of the limit?

$$\lim_{x \to 0+} \frac{\sin(5x)}{x}.$$

Enter a number: ______________________________

- Find the limit using a table. What are your commands? What is the estimated value? (You need values getting close to 3 not 0.)

$$\lim_{x \to 3} \frac{x^3 - 2x^2 - 9}{x^2 - 2x - 3}.$$

The commands are:

Your answer:

The value is:

Enter a number: ______________________________

- Find this limit using a table. What is the estimated value?

$$\lim_{x \to 0} \frac{\sin^{-1}(4x)}{\sin^{-1}(5x)}$$

Enter a number: ______________________________

- Find the *left* limit of `f(x) = cos(pi/2*(x - floor(x)))` as x goes to 2.

Enter a number: ______________________________

- Find the limit using a table. What is the estimated value? Recall, `atan` and `asin` are the names for the appropriate inverse functions.

$$\lim_{x \to 0+} \frac{\tan^{-1}(x) - 1}{\sin^{-1}(x) - 1}.$$

Enter a number: ______________________________

Symbolic limits

The add-on package `SymPy` can compute the limit of a simple algebraic function of a single variable quite well. The package is loaded when `MTH229` is.

`SymPy` provides the `limit` function. It is called just as our `lim` function is above. For example:

```
f(x) = sin(x)/x
limit(f, 0)
```

1

(This is a simplified form of the `limit` function, `SymPy` has more generality.)

- Find this limit using `SymPy` (use a decimal value for your answer, not a fraction):

$$\lim_{x \to 3} \frac{1/x - 1/3}{x^2 - 9}.$$

Enter a number: ______________________________

- Find this limit using `SymPy`:

$$\lim_{x \to 0} \frac{\sin(x^2)}{x \tan(x)}.$$

Enter a number: ______________________________

- Find the limit using `SymPy`. What is the estimated value?

$$\lim_{x \to 0+} \frac{x - \sin(|x|)}{x^3}.$$

Enter a number: ______________________________

Other questions

- Let `f(x) = sin(sin(x)^2) / x^k`. Consider $k = 1$, 2, and 3. For which of values of k does the limit at 0 **not** exist? (Repeat the problem for the 3 different values.)

Make a selection:

1. k=1,2,3
2. k=2,3
3. k=3
4. it exists for all k

- Let `l(x) = (a^x - 1)/x` and *define* $L(a) = \lim_{x \to 0} l(x, a)$.

Show that $L(3 \cdot 4) = L(3) + L(4)$ by computing all three limits numerically. (In general, you can show algebraically that $L(a \cdot b) = L(a) + L(b)$ like a logarithm. Show your work.

Your answer:

CHAPTER 6

Finding Derivatives

To get started, we load the `MTH229` package:

```
using MTH229
```

Quick background

Read about this material here: Approximate derivatives in julia.

For the impatient, A secant line connecting points on the graph of $f(x)$ between $x = c$ and $x = c + h$ has slope:

$$\frac{f(c+h) - f(c)}{h}.$$

The slope of the tangent line to the graph of $f(x)$ at the point $(c, f(c))$ is given by taking the limit as h goes to 0:

$$\lim_{h \to 0} \frac{f(c+h) - f(c)}{h}.$$

The notation for this - when the limit exists - is $f'(c)$.

In general the derivative of a function $f(x)$ is the function $f'(x)$, which returns the slope of the tangent line for each x where it is defined. For many functions, finding the derivative is straightforward, though my be complicated. At times approximating the value is desirable.

Approximate derivatives

We can approximate the slope of the tangent line several ways. The *forward difference quotient* takes a small value of h and uses the values $(f(x+h) - f(x))/h$ as an approximation.

For example, to estimate the derivative of x^x at $c = 1$ with `h=1e-6` we could have

```
f(x) = x^x
c, h = 1, 1e-6
(f(c+h) - f(c))/h
```

```
1.000001000006634
```

The above pattern finds the approximate derivative at the point c. Though this can be pushed to return a function giving the derivative at any point, we will use the more convenient solution described next for finding the derivative as a function, when applicable.

Automatic derivatives

In mathematics we use the notation $f'(x)$ to refer the function that finds the derivative of $f(x)$ at a given x. The `MTH229` package implements the same notation in `Julia`. (Though at the cost of a warning when the package is loaded.). This uses *automatic differentiation*, as provided by the `ForwardDiff` package, to compute. Automatic differentiation is a tad slower than using a hand-computed derivative, but as accurate and easier than using an *approximate derivative.* When available, automatic differentiation gives a convenient numeric value for the true derivative.

The usual notation for a derivative is used:

```
f(x) = sin(x)
f'(pi), f''(pi)
```

```
(-1.0,-1.2246467991473532e-16)
```

Symbolic derivatives

Automatic differentiation gives accurate numeric values for first, second, and even higher-order derivatives. It does not however, return the expression one would get were these computed by hand. The `diff` function from `SymPy` will find symbolic derivatives, similar to what is achieved when differentiating "by hand."

The `diff` function can be called with a function:

```
f(x) = exp(x) * sin(x)
diff(f)
```

$$e^x \sin{(x)} + e^x \cos{(x)}$$

A more general usage is supported, but not explored here.

Questions

- Calculate the slope of the secant line of $f(x) = 3x^2 + 5$ between $(2, f(2))$ and $(4, f(4))$.

Enter a number: ______________________________

- For the function $f(x) = 3x^2 + 5$ between $(2, f(2))$ and $(4, f(4))$ plot the function and the secant line. Estimate from the graph the largest distance between the two functions from x_0 to x_1.

Enter a number: ______________________________

- Consider the following `Julia` commands:

```
f(x) = sin(x)
sl(h) = (sin(pi/3 + h) - sin(pi/3)) / h
sl(0.1), sl(0.01), sl(0.001), sl(0.0001)
```

```
(0.45590188541076104,0.49566157577736871,0.49956690400077,0.4999566978958203)
```

These show what?

Make a selection:

1. The limit of ‘sin(x)‘ as ‘h‘ goes to ‘0‘ is ‘0.5‘
2. The limit of ‘sin(pi/3 + h)‘ as ‘h‘ goes to ‘0‘ is ‘0.5‘
3. The derivative of ‘sin‘ at ‘pi/3‘ is ‘1/2‘

- Let $f(x) = 1/x$ and $c = 4$. Find the approximate derivative (forward) when `h=1e-6`.

Enter a number: ______________________________

- Let $f(x) = x^x$ and $c = 4$. Find the approximate derivative (forward) when `h=1e-4`.

Enter a number: ______________________________

- For $f(x) = x^x$ and $c = 4$, use $f'(c)$ to find the numeric (automatic) derivative:

Enter a number: ____________________________

- Use the automatic derivative to find the slope of the tangent line at $x = 1/2$ for the graph of the function:

$$f(x) = log(\frac{1 + \sqrt{1 - x^2}}{x}) - \sqrt{1 - x^2}.$$

Enter a number: ____________________________

- Let $f(x) = \sin(x)$. Following the example on p124 of the Rogawski book we look at a table of values of the forward difference equation at $c = \pi/6$ for various values of h. The true derivative is $\cos(\pi/6) = \sqrt{3}/2$.

  Make the following table.

```
f(x) = sin(x)
c = pi/6
hs = [(1/10)^i for i in 1:12]
ys = [(f(c+h) - f(c))/h for h in hs] - sqrt(3)/2
[hs ys]
```

```
12x2 Array{Any,2}:
 0.1       -0.0264218
 0.01      -0.00251441
 0.001     -0.000250144
 0.0001    -2.50014e-5
 1.0e-5    -2.50002e-6
 1.0e-6    -2.49917e-7
 1.0e-7    -2.51525e-8
 1.0e-8    -2.39297e-9
 1.0e-9     1.42604e-8
 1.0e-10    1.80794e-7
 1.0e-11   -1.48454e-6
 1.0e-12    4.06657e-6
```

  What size `h` has the closest approximation?

Make a selection:

1. 1e-1
2. 1e-2
3. 1e-3

4. 1e-4
5. 1e-5
6. 1e-6
7. 1e-7
8. 1e-8
9. 1e-9
10. 1e-10
11. 1e-11
12. 1e-12

- For the same $f(x) = \sin(x)$ and $c = \pi/6$, how accurate is the automatic derivative found with `f'`?

Enter a number: ______________________________

- Let $f(x) = (x^3 + 5) \cdot (x^3 + x + 1)$. The derivative of this function has one real zero. Find it. (You can use `fzero` with the derivative function after plotting to identify a bracketing interval.)

Enter a number: ______________________________

- Make a plot of $f(x) = \log(x+1) - x + x^2/2$ and its derivative over the interval $[-3/4, 4]$. The commands are:

Your answer:

Is the derivative always increasing?

Make a selection:

1. Yes
2. No

- Let $f(x) = (x+2)/(1+x^3)$. Plot both f and its derivative on the interval $[0,5]$. Identify the zero of the derivative. What is its value? What is the value of $f(x)$ at this point?

What commands produce the plot?

Your answer:

What is the zero of the derivative on this interval?

Enter a number: ____________________________

What is the value of f at this point:

Enter a number: ____________________________

- The function $f(x) = x^x$ has a derivative for $x > 0$. Use `fzero` to find a zero of its derivative. What is the value of the zero?

Enter a number: ____________________________

- Using the `diff` function from the `SymPy` package, identify the proper derivative of x^x:

Make a selection:

1. $x \cdot x^{(x-1)}$
2. $x^x \cdot (\log(x) + 1)$
3. $x^{(x+1)}/(x + 1)$
4. x^x

Letting $c = 4$, we can find how accurate `f'(c)` is for $f(x) = x^x$ by using the expression found in the last answer evaluated at c and taking the difference with `f(c)`. How big is the difference?

Enter a number: ____________________________

- Using the `diff` function, find the derivative of the inverse tangent, $\tan^{-1}(x)$ (`atan`). What is the function?

Make a selection:

1. $1/(x^2 + 1)$
2. $(-1) \cdot \tan^{-2}(x) \cdot (\tan^2(x) + 1)$
3. $(-1) \cdot \tan^{-2}(x)$

Some applications

- Suppose the height of a ball falls according to the formula $h(t) = 300 - 16t^2$. Find the rate of change of height at the instant the ball hits the ground.

Enter a number: ____________________________

- A formula for blood alcohol level in the body based on time is based on the number of drinks and the time wikipedia.

Suppose a model for the number of drinks consumed per hour is

```
n(t) = t <= 3 ? 2 * sqrt(3) * sqrt(t) : 6.0
```

```
n (generic function with 1 method)
```

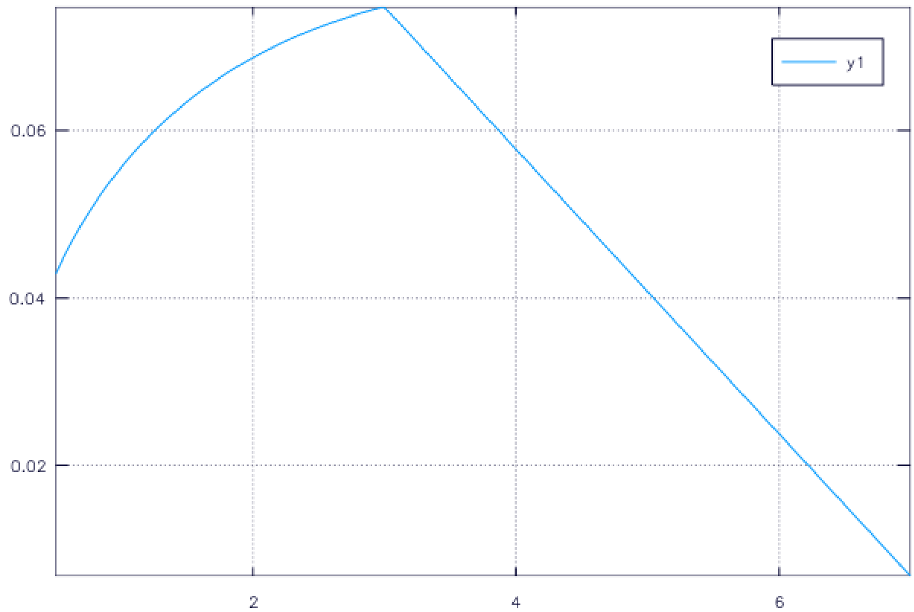
Then the BAL for a 175 pound male is given by

```
bal(t) = (0.806 * 1.2 * n(t)) / (0.58 * 175 / 2.2) - 0.017*t
```

```
bal (generic function with 1 method)
```

From the plot below, describe when the peak blood alcohol level occurs and is the person ever in danger of being above 0.10?

```
plot(bal, .5,7)
```



Your answer:

- Plot the derivative of `bal` over the time $[0.5, 7]$. Is this function ever positive?

Make a selection:

1. Yes, after 3

2. Yes, initially

3. No, it never is

Tangent lines

The tangent line to the graph of $f(x)$ at $x = c$ is given by $y = f(c) + f'(c)(x - c)$. It is fairly easy to plot both the function and its tangent line - we just need a function to compute the tangent line.

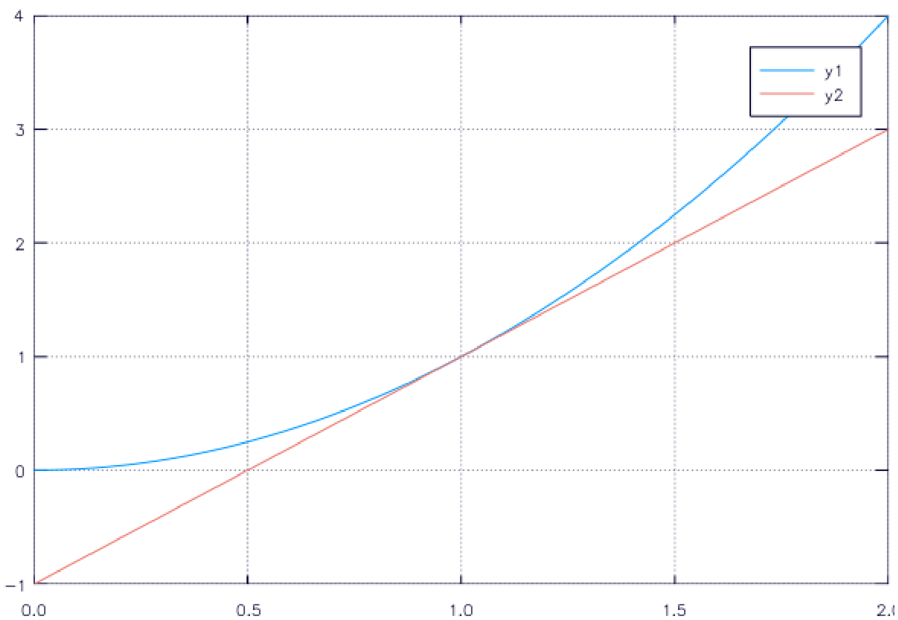
Here we write an operator to return such a function. The operator needs to know both the function name and the value c to find the tangent line at $(c, f(c))$ (notice the `x->` bit indicating the following returns a function):

```
tangent(f, c) = x -> f(c) + f'(c)*(x-c) # returns a function
```

(This function is in the `MTH229` package.)

Here we see how to use it:

```
f(x) = x^2                         # replace me
plot(f, 0, 2)
plot!(tangent(f, 1), 0, 2)
```



- For the function $f(x) = 1/(x^2 + 1)$ (The witch of Agnesi), graph f over the interval $[-3, 3]$ and the tangent line to f at $x = 1$. The tangent line intersects the graph at $x = 1$, where else?

Enter a number: ____________________________

- Let $f(x) = x^3 - 2x - 5$. Find the intersection of the tangent line at $x = 3$ with the x-axis.

Enter a number: ____________________________

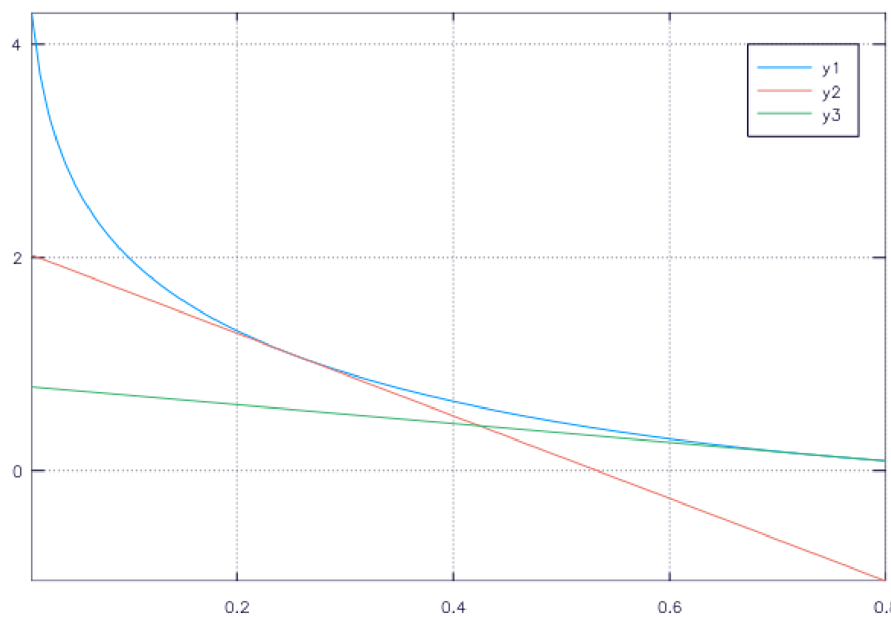
- Let $f(x)$ be given by the expression below.

```
f(x; a=1) = a * log((a + sqrt(a^2 - x^2))/x ) - sqrt(a^2 - x^2)
```

```
f (generic function with 1 method)
```

The value of `a` is a parameter, the default value of $a = 1$ is fine.
For $x = 0.25$ and $x = 0.75$ the tangent lines can be drawn with

```
u, v = 0.01, 0.8
plot(f, u, v)
plot!(tangent(f, 0.25), u, v)
plot!(tangent(f, 0.75), u, v)
```



Verify that the length of the tangent line between $(c, f(c))$ and the y axis is the same for $c = 0.25$ and $c = 0.75$. (For any c, the distance formula can be used to find the distance between the point $(c, f(c))$ and $(0, y_0)$ where, y_0 is where the tangent line at c crosses the y axis.)

Your answer:

Higher-order derivatives

Higher-order derivates can be approximated as well. For example, one can use `f''` to approximate the second derivative.

- Find the second derivative of $f(x) = \sqrt{x \cdot e^x}$ at $c = 2$.

Enter a number: ____________________________

- Find the zeros in $[0, 10]$ of the second derivative of the function $f(x) = \sin(2x) + 3\sin(4x)$ using `fzeros`.

Make a selection:

1. 13 numbers: 0.0, 0.806238, ..., 8.61854, 9.42478
2. 13 numbers: 0.420534, 1.20943, ..., 9.00424, 9.84531
3. 13 numbers: 0.0, 0.869122, ..., 8.55566, 9.42478

CHAPTER 7

The First and Second Derivative Properties

Exploring first and second derivatives with Julia:

To get started, we load the `MTH229` package:

```
using MTH229
```

Recall, the `MTH229` package overloads `'` so that the same prime notation of mathematics is available in `Julia` for indicating derivatives of functions.

Quick background

Read about this material here: Exploring first and second derivatives with Julia.

For the impatient, this assignment looks at the relationship between a function, $f(x)$, and its first and second derivatives, $f'(x)$ and $f''(x)$. The basic relationship can be summarized (though the devil is in the details) by:

- If the first derivative is *positive* on (a, b) then the function is *increasing* on (a, b).
- If the second derivative is *positive* on (a, b) then the function is *concave up* on (a, b).

(The "devil" here is that the converse statements are usually - but not always - true.)

As a reminder

- A **critical** point of f is a value in the domain of $f(x)$ for which the derivative is 0 or undefined. These are often - but **not always** - where $f(x)$ has a local maximum or minimum.
- An **inflection point** of f is a value in the domain of $f(x)$ where the concavity of f changes. (These are *often* - but **not always** - where $f''(x) = 0$.)

In addition, there are two main derivative tests:

- The **first derivative test**: This states that for a differentiable function $f(x)$ with a critical point at c then if $f'(x)$ changes sign from $+$ to $-$ at c then $f(c)$ is a local maximum and if $f'(x)$ changes sign from $-$ to $+$ then $f(c)$ is a local minimum.
- The **second derivative test**: This states that if c is a critical point of $f(x)$ and $f''(c) > 0$ then $f(c)$ is a local minimum and if $f''(c) < 0$ then $f(c)$ is a local maximum.

To investigate these concepts in `Julia` we describe a few functions.

In the notes, the following function is used to plot a function `f` using two colors depending on whether the second function, `g` is positive or not. This function is in the `MTH229` package.

```
function plotif(f, g, a, b)
  plot([f, x -> g(x) > 0.0 ? f(x) : NaN], a, b, linewidth=5)
end
```

This function allows a graphical exploration of the above facts. For example, `plotif(f,f,a,b)` will show a different color when $f(x)$ is *positive* and `plotif(f,f', a, b)` will show a different color when $f(x)$ is *increasing*. The latter is illustrated with:

```
f(x) = 1 + cos(x) + cos(2x)
plotif(f, f', 0, 2pi)  # color increasing
plot!(zero)
```

```
ErrorException("If you want to plot the function ‘zero‘, you need to define the x values!")
```

We can graphically identify zeros or critical points or find them numerically by finding zeroes of the appropriate function. To find zeros we have the function call `fzero(f, a)` to find a zero iteratively starting at $x = a$ or `fzeros(f, a, b)` to naively search for any zeros in the interval $[a, b]$. (Recall, `fzeros` may miss some values, so a graph should always be made to double check)

For example to find a zero in `f` near `1.5`:

```
f(x) = 1 + cos(x) + cos(2x)
fzero(f, 1.5)
```

```
1.5707963267948966
```

Or to get any critical points of `f` (it is continuosly differentiable, so all critical points are given by solving $f'(x) = 0$):

```
zs = fzeros(f', 0, 2pi)
```

```
5-element Array{Float64,1}:
 0.0
 1.82348
 3.14159
 4.45971
 6.28319
```

The answer from `fzeros` is a vector of values. You can get individual ones different ways or work with them all at once. For example, here is the function's value at each of the critical points:

```
map(f, zs)
```

```
5-element Array{Float64,1}:
  3.0
 -0.125
  1.0
 -0.125
  3.0
```

Questions

Graphical explorations

- The `airy` function is a *built-in* function that is important for some applications. It is likely to be unfamiliar. Make a graph using `plotif` to investigate when the `airy` function is positive on the interval $(-5, 5)$. Your answer should use interval notation. (Recall, when the second function passed to `plotif` is positive, the graph uses a different color, so you need to think about what function that should be.)

  Your answer:

- Make a graph using `plotif` to investigate when the function $f(x) = x^x$ is *increasing* on the interval $(0, 2)$. Your answer should use interval notation.

Your answer:

- Make a graph using `plotif` to investigate when the function

$$f(x) = \frac{x}{x^2 + 9}$$

is *concave up* on the interval $(-10, 10)$. Your answer should use interval notation.

Your answer:

- Make a graph using `plotif` to identify any *critical points* of $f(x) = x\ln(x)$ on the interval $(0, 4)$. Points where the function changes from increasing to decreasing will be critical points (though there may be others).

Your answer:

- Make a graph using `plotif` to identify any *inflection points* of $f(x) = \sin(x) - x$ over the interval $(-5, 5)$. Points where the function changes concavity are inflection points (though there may be others).

Your answer:

- For any polynomial $p(x)$, between any two consecutive zeros there must be a critical point, perhaps more than one.

For $p(x) = x^4 + x^3 - 7x^2 - x + 6$, there are zeros -3, -1, 1 and 2. Using `plotif`, identify which critical point(s) are in $[-1, 1]$?

Make a selection:

1. -0.07046
2. $-0.72, 0.75$
3. 0
4. $0, 0.25, 0.57$
5. There are no critical points, as p(x) is not 0 in (-1,1)

Finding more precise numeric values

- Use `fzero` or `fzeros` to numerically identify all *critical points* to the function $f(x) = 2x^3 - 6x^2 - 2x + 4$. (There are no more than 2.)

Make a selection:

1. $-0.860806, 0.745898, 3.11491$

2. $0.745898, 3.11491$
3. $-0.154701, 2.1547$
4. 1

- Use `fzero` of `fzeros` to numerically identify all *inflection points* for the function $f(x) = \ln(x^2 + 2x + 5)$.

Make a selection:

1. There are none
2. There is one at $x = -1.0$
3. There is one at $x = 1.0$ and one at $x = -3.0$
4. There is one at each of $x = -4.4641$, -1.0, and 2.4641

- Numerically identify all *critical points* to the rational function $f(x)$ defined below. Graphing is useful to identify where the possible values are. (A critical point by definition is in the domain of the function.)

$$f(x) = \frac{(x-3)\cdot(x-1)\cdot(x+1)\cdot(x+3)}{(x-2)\cdot(x+2)}.$$

Make a selection:

1. $-3, -1, 1, 3$
2. 0
3. $-2, 2$
4. $-2.44949, 2.44949$

- Suppose the first derivative of f is $f'(x) = x^3 - 6x^2 + 11x - 6$. Where is $f(x)$ increasing? Use interval notation in your answer.

Make a selection:

1. It is always increasing

2. $(2.0, \infty)$
3. $(-\infty, 1.42265)$ and $(2.57735, \infty)$
4. $(1.0, 2.0)$ and $(3.0, \infty)$

- Suppose the second derivative of f is $f''(x) = x^2 - 3x + 2$. Where is $f(x)$ concave up? Use interval notation in your answer.

Make a selection:

1. $(-\infty, \infty)$ – it is always concave up
2. $(1.5, \infty)$
3. $(1.0, 2.0)$
4. $(-\infty, 1.0)$ and $(2.0, \infty)$

- For the function $f(x)$ suppose you know $f'(x) = x^3 - 5x^2 + 8x - 4$. Find *all* the critical points. Use the first derivative test to classify them as local extrema *if* you can. If you can't say why.

Your answer:

- Suppose the first derivative of f is $f'(x) = (x^2 - 2) \cdot e^{-x}$. First find the critical points of $f(x)$, then use the second derivative test to classify them.

The critical points are:

Make a selection:

1. $-0.732051, 2.73205$
2. -0.732051
3. 0.0
4. $-1.41421, 1.41421$

Classify your critical points using the second derivative test

Your answer:

- Suppose the first derivative of f is $f'(x) = x^3 - 7x^2 + 14$. Based on a plot over the interval $[-4, 8]$. On what subintervals is $f(x)$ increasing?

Make a selection:

1. $(-\infty, 0)$
2. $(-1.29, 1.61)$ and $(6.69, \infty)$
3. $(-\infty, 0)$ and $(4.67, \infty)$
4. $(-\infty, 0)$ and $(6.69, \infty)$

What did you use to find your last answer?

Make a selection:

1. $f'(x) > 0$ on these subintervals
2. $f''(x) > 0$ on these subintervals
3. $f'(x) < 0$ on these subintervals
4. $f''(x) < 0$ on these subintervals

What are the x-coordinates of the relative minima of $f(x)$?

Make a selection:

1. 4.56
2. 4.56 and 0
3. -1.29 and 1.61
4. -1.29 and 6.69

On what subintervals is $f(x)$ concave up?

Make a selection:

1. $(1.167, \infty)$
2. $(-\infty, 1.167)$
3. $(-\infty, 0)$ and $(4.67, \infty)$
4. It is always concave down

What did you use to decide?

Make a selection:

1. $f'(x) > 0$ on these subintervals
2. $f''(x) > 0$ on these subintervals
3. $f'(x) < 0$ on these subintervals
4. $f''(x) < 0$ on these subintervals

Find the x coordinates of the inflection points of $f(x)$.

Make a selection:

1. 2.3333
2. At 0 and 4.67
3. Not listed
4. $-1.29884, 1.61194, 6.6869$

- Suppose you know the function $f(x)$ has the second derivative given by the `airy` function. Use this to answer the following questions about $f(x)$ over the interval $(-5, 0)$.

On what interval(s) is the function $f(x)$ positive?

Make a selection:

1. $(-5, -4.08795)$ and $(-2.33811, 0)$
2. $(-5, -4.8201)$ and $(-3.2482, -1.01879)$
3. $(-4.83074, -3.27109)$ and $(-1.17371, 0)$
4. Can't tell.

On what interval(s) is the function $f(x)$ increasing?

Make a selection:

1. $(-5, -4.08795)$ and $(-2.33811, 0)$
2. $(-5, -4.8201)$ and $(-3.2482, -1.01879)$
3. $(-4.83074, -3.27109)$ and $(-1.17371, 0)$
4. Can't tell.

On what interval(s) is the function $f(x)$ concave up?

Make a selection:

1. $(-5, -4.08795)$ and $(-2.33811, 0)$
2. $(-5, -4.8201)$ and $(-3.2482, -1.01879)$
3. $(-4.83074, -3.27109)$ and $(-1.17371, 0)$
4. Can't tell.

- A simplified model for the concentration (micrograms/milliliter) of a certain slow-reacting antibiotic in the bloodstream t hours after injection into muscle tissue is given by

$$f(t) = t^2 \cdot e^{-t/16}, \quad t \geq 0.$$

When will there be maximum concentration?

Enter a number: ______________________________

In the units given, how much is the maximum concentration?

Enter a number: ____________________________

When will the concentration dip below a level of 20.0?

Enter a number: ____________________________

Estimate from a graph when the concentration function changes concavity:

Your answer:

- (From Rogawski) Ornithologists have found that the power consumed ($Joules/sec$) by a bird flying a certain velocity is given (in Joules) by

$$P(v) = \frac{16}{v} + (\frac{v}{10})^3.$$

A bird stores $5 \cdot 10^4$ joules of energy, so the total distance it can fly at a fixed velocity v depends on the velocity and is given by (rate times time):

$$D(v) = v \cdot \frac{5 \cdot 10^4}{P(v)}.$$

- Find the velocity v that minimizes $P(v)$. It happens at the critical point.

Enter a number: ____________________________

- Migrating birds are actually a bit smarter and can adjust their velocity to *maximize* distance traveled, and not *minimize* power consumed. Find the velocity that maximizes D. It happens at a critical point.

Enter a number: ______________________________

- Let v_d be the velocity that *maximizes* distance. What is the value of

$$P'(v_d) - \frac{P(v_d)}{v_d}?$$

Enter a number: ______________________________

CHAPTER 8

Newton-Raphson Method

Begin by loading our package that brings in plotting and other features, including those provided by the `Roots` package:

```
using MTH229
```

Quick background

Read about this material here: Newton's Method.

For the impatient, symbolic math - as is done behind the scenes at the Wolfram alpha web site - is pretty nice. For so many problems it can easily do what is tedious work. However, for some questions, only numeric solutions are possible. For example, there is no general formula to solve a fifth order polynomial the way there is a quadratic formula for solving quadratic polynomials. Even an innocuous polynomial like $f(x) = x^5 - x - 1$ has no easy algebraic solution.

Numeric solutions are available. As this is a polynomial, we could use the `roots` function from the `Roots` package:

```
f(x) = x^5 - x - 1
roots(f)
```

```
5-element Array{Complex{Float64},1}:
     1.1673+0.0im
   0.181232+1.08395im
   0.181232-1.08395im
  -0.764884+0.352472im
  -0.764884-0.352472im
```

We see 5 roots, as expected from a fifth degree polynomial, with one real root (the one with `0.0im`) that is approximately 1.1673. Finding such a value usually requires some iterative root-finding algorithm (though not in the case above which uses linear algebra). For polynomials, the **fzeros** function uses such an algorithm *for polynomials* to find the real roots:

```
fzeros(f)                 # no a, b range needed for polynomials.
```

```
1-element Array{Real,1}:
 1.1673
```

However, more general techniques are needed for non-polynomials. We've seen the bisection method previously to find a root, but this is somewhat cumbersome to use as it needs a bracketing interval to begin.

Here we discuss Newton's method. Like the bisection method it is an *iterative algorithm*. However instead of identifying a bracketing interval, we only need to identify a reasonable *initial* guess, x_0.

Starting with x_0 the algorithm to produce x_1 is easy:

- form the tangent line at $(x_0, f(x_0)$.

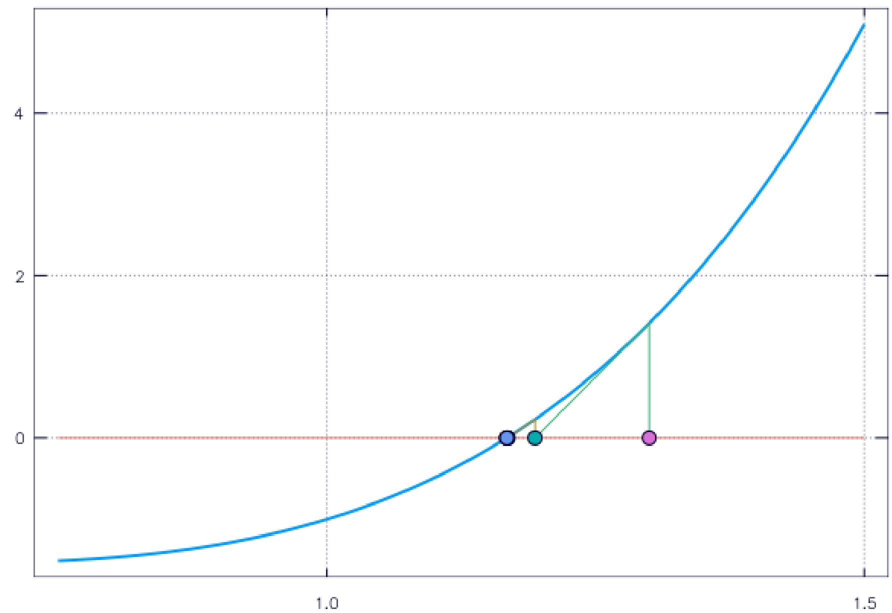
- let x_1 be the intersection point of this tangent line:

If we can go from x_0 to x_1 we can repeat to get x_2, x_3, ...

Graphically, the `MTH229` package provides a *helper function* - `newton_viz` - to illustrate the process:

```
f(x) = x^5 - x - 1
newton_viz(f, 1.3)
```

```
Signal{Plots.Plot{Plots.GRBackend}}(Plot{Plots.GRBackend() n=5}, nactions=0)
```



In the figure, the sequence of guesses can be seen, basically 1.3, 1.19, 1.168, ...

To find these numerically, we first need an algebraic representation. For this problem, we can describe the tangent line's slope by *either* $f'(x_0)$ *or* by using "rise over run":

$$f'(x_0) = \frac{f(x_0) - f(x_1)}{x_0 - x_1}$$

Using $f(x_1) = 0$, this yields the update formula: $x_1 = x_0 - f(x_0)/f'(x_0)$. That is, the new guess shifts the old guess by an increment $f(x_0)/f'(x_0)$.

In `Julia`, we can do one step with:

```
f(x) = x^5 - x - 1
x = 1.3
x = x - f(x) / f'(x)
```

```
1.1936086743721999
```

(We don't use indexing, but rather update our binding for the `x` variable.)

Is `x` close to being the zero? We don't know the actual zero - we are trying to approximate it - but we do know the function's value at the actual zero. For this new guess the function value is

```
f(x)
```

```
0.2291481834425222
```

This is much closer than $f(1.3)$, the value at our initial guess, but not nearly as close as we can get using Newton's method. We just need to **iterate** - run a few more steps.

We do another step just by running the last line. For example, we run 4 more steps by copying and pasting the same expression:

```
x = x - f(x) / f'(x)
x = x - f(x) / f'(x)
x = x - f(x) / f'(x)
x = x - f(x) / f'(x)
```

```
1.1673039782614187
```

The value of `x` updates. But is it getting closer to a *zero*? If so, then $f(x)$ should be close to zero. We can see both values with:

```
x, f(x)
```

```
(1.1673039782614187,6.661338147750939e-16)
```

This shows $f(x)$ is not exactly 0.0 but it is as close as we can get. Repeating the algorithm does not change the value of `x`. (On a computer, floating point issues creep in when values are close to 0, and these prevent values being mathematically exact.) As we can't improve, we stop. Our value of `x` is an *approximate* zero and `f(x)` is within machine tolerance of being `0`.

How do we know how to stop? When the algorithm works, we will stop when the `x` value *basically* stops updating, as `f(x)` is basically `0`. However, the algorithm need not work, so any implementation must keep track of how many steps are taken and stop when this gets out of hand.

For convenience, the `newton` method from the `Roots` package will iterate until convergence. If we pass in the optional argument `verbose=true` we will see the sequence of steps.

For example, for $f(x) = x^3 - 2x - 5$, a function that Newton himself considered, a solution near 2, is found with:

```
x = 2
f(x) = x^3 - 2x -5
xstar = newton(f, 2)
xstar, f(xstar)
```

```
(2.0945514815423265,-8.881784197001252e-16)
```

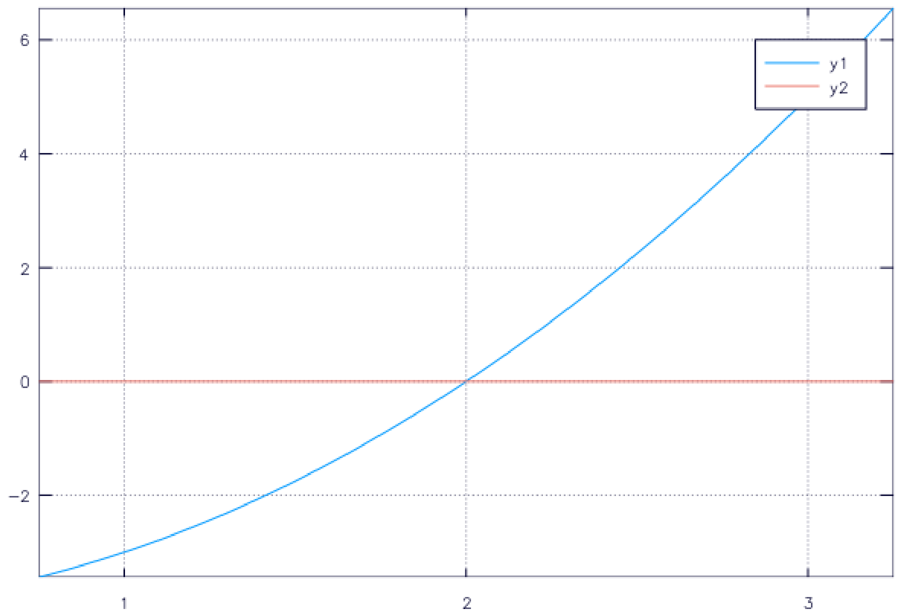
Questions

- Use `newton_vis` to visualize the first 5 steps of Newton's method (the default) for solving for $f(x) = x^2 - 2$ starting at 2. From the graph, estimate the value after the *first* step:

Enter a number: ____________________________

- The function $x^2 - 4$ has a zero at $x^* = 2$, of course. Here is a graph:

```
plot(x -> x^2 - 4, .75, 3.25)
plot!(zero, .75, 3.25)
```



For such-shaped graphs (increasing, concave up) some things will always be the case. If Newton's method is started to the *left* of x^*, say at $x_0 = 1$ then:

Make a selection:

1. The value of x_1 will be to the left of x_0
2. The value of x_1 will be between x_0 and x^*.
3. The value of x_1 will be more than x^*

If Newton's method is started to the *right* of x^*, say at $x_0 = 3$ then:

Make a selection:

1. The value of x_1 will be to the right of x_0
2. The value of x_1 will be between x^* and x_0.
3. The value of x_1 will be to the left than x^*

- Apply Newton's Method to the function $f(x) = \sin(x)$ with an initial guess 3. (This was historically used to compute many digits of π efficiently.) What is the answer after 2 iterations? What is the value of `sin` at the answer?

The value of x after 2 iterations is:

Enter a number: ______________________________

The value of $f(x) = \sin(x)$ after 2 iterations:

Enter a number: ______________________________

- Use Newton's method to find a zero for the function $f(x) = x^5 - x - 1$. Start at $x = 1.6$. What is the approximate root after 5 iterations? What is the value of $f(x)$ for your answer? If you do one or two more iterations, will your guess be better?

The value of x_5 (after 5 iterations):

Enter a number: ______________________________

The value of $f(x_5)$ (after 5 iterations)? Choose the closest.

Make a selection:

1. 1.2e-6

2. 1.2e-8

3. 1.2e-10

4. 1.2e-12

5. 1.2e-14

6. 1.2e-16

The value of x_7 (after *two* more iterations):

Enter a number: ______________________________

- Use Newton's method, through the `newton` function, to find a zero of the function $f(x) = \cos(x) - 6x$. Make a graph to identify an initial guess.

Show your commands below

Your answer:

The value of the approximate zero:

Enter a number: ____________________________

- Use Newton's method to find an intersection point of $f(x) = e^{-x^2}$ and $g(x) = x$. (Look at $h(x) = f(x) - g(x) = 0$.) Start with a guess of 0.

Enter a number: ____________________________

- Use Newton's method to find *both* positive intersection points of $f(x) = e^x$ and $g(x) = 2x^2$. Make a graph to identify good initial guesses. (You need to use Newton's method twice, each with different initial guesses.)

The smallest *positive* value is:

Enter a number: ____________________________

The largest *positive* value is:

Enter a number: ____________________________

- The function `f(x) = exp(x) / (1 + 2exp(x))` has an inflection point near -0.5. Use Newton's method to find it. The inflection point occurs at $x = ...$:

Enter a number: ____________________________

Using fzero from the Roots package

As mentioned, the `newton` function in the `Roots` package implements Newton's method. The `Roots` package also provides the `fzero` function for finding roots. We have seen it used with a bracketing interval, but it also provides a solution when just given an initial guess - like Newton's method:

```
fzero(sin, 3)   # start with initial guess of 3
```

```
3.141592653589793
```

The utility of this function is that it does not require a derivative to be taken and it is a little less sensitive than Newton's method to the initial guess. However, it can involved many more function calls, so can be slower.

- find a zero of $f(x) = x \cdot (2 + \ln(x))$ starting at 1. What is your answer? How small is the function for this value?

What is the value of the zero?

Enter a number: ___________________________

The value of the function at the zero?

Enter a number: ___________________________

- Use `fzero` to find when the derivative of $f(x) = 5/\cos(x) + 7/\sin(x)$ is 0 in the interval $(0, \pi/2)$.

Enter a number: ___________________________

- The function $f(x) = x^5 - x^4 - x^3 - x^2 - x - 1$ has a real zero near 2. Find it:

Enter a number: ___________________________

The same function has a critical point in $[-2, 1]$. Make graph to find a good approximation, then use `fzero` or Newton's method to find a more exact value for this:

Enter a number: ___________________________

When Newton's method fails

The error in Newton's method at a simple zero follows a formula: $|e_{n+1}| \leq (1/2)f''(a)/f'(b)|e_n|^2$, for some a and b. Generally this ensures that the error at step $n + 1$ is smaller than the error at step n squared. But this can fail due to various cases:

- the initial guess is not close to the zero
- the derivative, $|f'(x)|$, is too small
- the second derivative, $|f''(x)|$, is too big, or possibly undefined.

- Earlier the roots of $f(x) = x^5 - x - 1$ were considered. Try Newton's method with an initial guess of $x_0 = 0$ to find a real root. Use `newton_vis` to visualize the output. Why does this fail? (You can look graphically. Otherwise, you could look at the output of `newton` with this extra argument: `newton(f, fp, x0, verbose=true)`.

Make a selection:

1. The initial guess is not close to the zero
2. The derivative, $|f'(x)|$, is too small
3. The second derivative, $|f''(x)|$, is too big, or possibly undefined

- Let `f(x) = abs(x)^(1/3)`. Starting at `x=1`, Newton's method will fail to converge. What happens? Are any of the above 3 reason's to blame?

Make a selection:

1. The initial guess is not close to the zero
2. The derivative, $|f'(x)|$, is too small
3. The second derivative, $|f''(x)|$, is too big, or possibly undefined

- Let $f(x) = x^2 - 0.01$. Though the initial value $x = 0$ is very close to a zero, Newton's method will fail when started there. Why?

Make a selection:

1. The initial guess is not close to the zero
2. The derivative, $|f'(x)|$, is too small
3. The second derivative, $|f''(x)|$, is too big, or possibly undefined

Quadratic convergence

When Newton's method converges to a *simple zero* it is said to have *quadratic convergence*. A simple zero is one with multiplicity 1 and quadratic convergence says basically that the error at the $i + 1$st step is like the error for ith step squared. In particular, if the error is like 10^{-3} on one step, it will be like 10^{-6}, then 10^{-12} then 10^{-24} on subsequent steps. (Which is typically beyond the limit of a floating point approximation.) This is why one can *usually* take just 5, or so, steps to get to an answer.

Not so for multiple roots and some simple roots.

- For the function `f(x) = (8x*exp(-x^2) -2x - 3)^8`, starting with `x=-2.0`, Newton's method will converge, but it will take many steps to get to an answer that has $f(x)$ around 10^{-16}. How many? Roughly how many iterations do you need? (Use `verbose=true` with `newton` to see.)

Make a selection:

1. about 5 steps
2. about 10 steps
3. about 15 steps
4. about 20 steps
5. about 25 steps
6. about 30 steps
7. about 35 steps
8. about 40 steps

- Repeat the above with `f(x) = 8x*exp(-x^2) -2x - 3` - there is no extra power of 8 here - and again, starting with `x=-2.0`. Roughly how many iterations are needed now?

Make a selection:

1. about 5 steps
2. about 10 steps
3. about 15 steps
4. about 20 steps
5. about 25 steps
6. about 30 steps
7. about 35 steps
8. about 40 steps

- The value 1 is a simple zero of $f(x) = x^{20} - 1$. A theorem can show that, in theory, Newton's method will converge for any starting point $x_0 > 0$. However, it can take awhile. How many steps does Newton's method take starting at 0.8?

Enter a number: ____________________________

CHAPTER 9

Optimization Problems

To get started, we load our `MTH229` package for plotting and other features:

```
using MTH229
```

Quick background

Read about this material here: Maximization and minimization with julia.

For the impatient, *extrema* is nothing more than a fancy word for describing either a maximum *or* a minimum. In calculus, we have **two** concepts of these: *relative* extrema and *absolute* extrema. Let's focus for a second on *absolute* extrema which are stated as:

> A value $y = f(x)$ is an *absolute maximum* over an interval $[a, b]$ if $y \geq f(x)$ for all x in $[a, b]$. (An absolute minimum has $y \leq f(x)$ instead.)

Of special note is that an absolute extrema involves *both* a function **and** an interval.

There are two theorems which help identify extrema here. The first, due to Bolzano, says that any continuous function on a *closed* interval will *necessarily* have an absolute maximum and minimum on that interval. The second, due to Fermat, tells us where to look: these absolute maximums and minimums can only occur at end points or critical points.

Bolzano and Fermat are historic figures. For us, we can plot a function to visually see extrema. The value of Bolzano is the knowledge that yes, plotting isn't a waste of time, as we are *guaranteed* to see what we look for. The value of Fermat is that if we want to get *precise* numeric answers, we have a means: identify the end points and the critical points then compare the function at *just* these values.

The notes walk us through the task of finding among all rectangles with perimeter 20 the one with maximum area.

This leads to two equations:

- a constraint based on a fixed perimeter: $20 = 2b + 2h$.
- an expression for the area: $A = h \cdot b$.

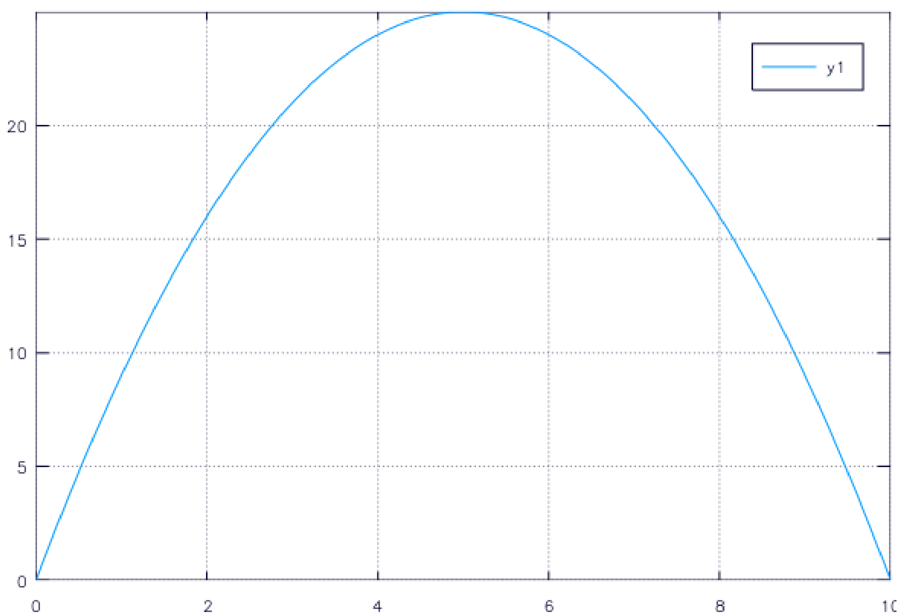
We translate these into `Julia` functions. First, using the constraint, we solve for one variable and then substitute this in:

```
h(b) = (20 - 2b) / 2
A(b) = h(b) * b    # A = h * b translates to this
```

```
A (generic function with 1 method)
```

So we have area as a function of a single variable. Here `b` ranges from 0 to no more than 10, as both `b` and `h` need be non-neagative. A plot shows the function to maximize:

```
plot(A, 0, 10)
```



Not only do we see a maximum value, we also can tell more:

- the maximum happens at a critical point - not an end point
- there is a unique critical point on this interval $[0, 10]$.

So, we can use `fzero` to find the critical point:

```
x = fzero(A', 5)
```

```
5.0
```

We store the value as `x`. Is this the answer? Not quite, the question asks for the rectangle that gives the maximum area, so we should also find the height. But this is just

```
h(x)
```

```
5.0
```

In fact, for the problems encountered below, the critical point, the constraint at the critical point, or the function evaluated at the critical point are used to answer the questions:

```
x, h(x), A(x)
```

```
(5.0,5.0,25.0)
```

Questions

For the following questions (which were cribbed from various internet sources) find the most precise answer available, a graphical solution is not enough, the answers should use one of the zero-finding methods.

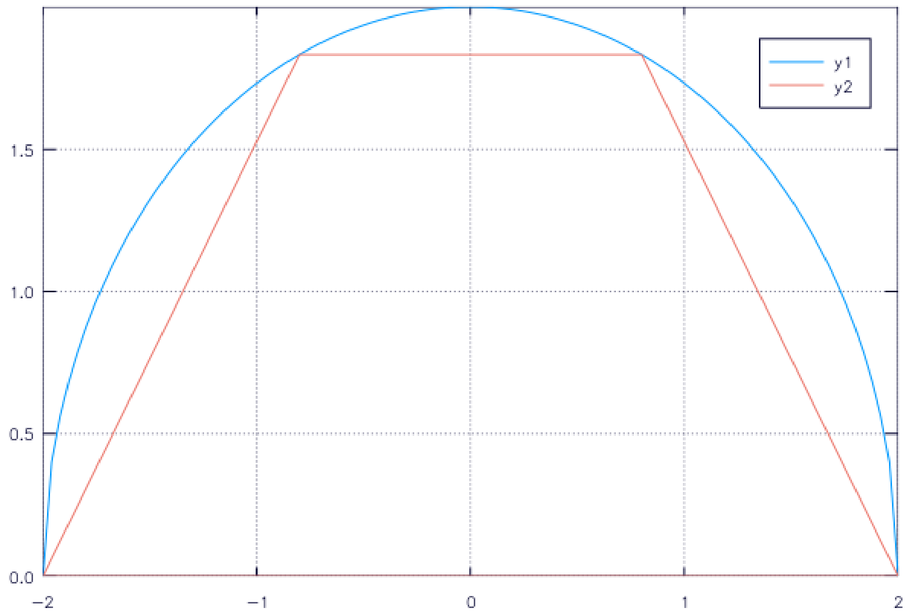
- *Ye olde post office*

A box with a square base is taller than it is wide. In order to send the box through the U.S. mail, the height of the box and the perimeter of the base can sum to no more than 108 inches. Show how to compute the maximum volume for such a box.

Your answer:

- *Inscription*

A trapezoid is inscribed in a semicircle of radius $r = 2$ so that one side is along the diameter. Find the maximum possible area for the trapezoid.



The area of trapezoid is the height times the average of the two bases or with this picture: $(2r + 2x)/2 \cdot y$.

Your answer:

- *Cheap paper cups*

A cone-shaped paper drinking cup is to hold 100 cubic centimeters of water (about 4 ozs). Find the height and radius of the cup that will require the least amount of paper. The volume of such a cup is given by the volume of a cone formula: $V = (1/3)\pi r^2 h$; and the area of the paper is given by for formula for surface area of a cone: $A = \pi r \sqrt{r^2 + h^2}$. Show any work.

Your answer:

- *How big is that can?*

A cylindrical can, **open on top**, is to hold 355 cubic centimeters of liquid. Find the height and radius that minimizes the amount of material needed to manufacture the can. (These are metric units, so the answer will be in centimeters with 2.54cm=1in.) Illustrate how this is done:

Your answer:

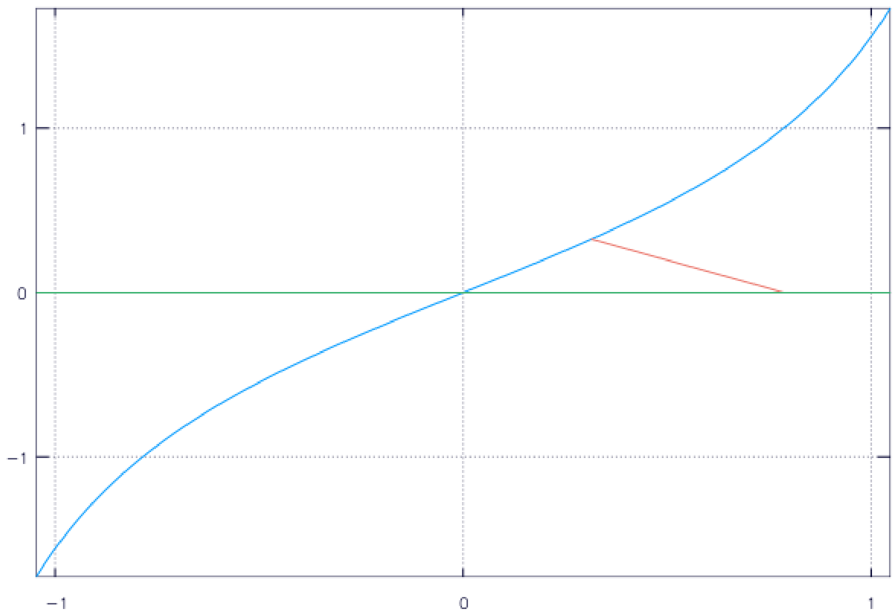
Do these proportions match those that are typical for a 12 oz can?

Make a selection:

1. Yes, the height is about 2 times the diameter
2. No, the can has a square profile
3. No, the diameter is twice the height

- *Getting closer*

Let $f(x) = \tan(x)$. Find the point on the graph of $f(x)$ that is closest to the point $(\pi/4, 0)$. Show any work.



Recall the distance formula: $d = \sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2}$, as we need to minimize this.

Your answer:

- *Best size for a phone*

A cell phone manufacturer wishes to make a rectangular phone with total surface area of 12,000 mm^2 and maximal screen area. The screen is surrounded by bezels with sizes of $8mm$ on the long sides and $32mm$ on the short sides. (So the width of the screen would be the width of the phone minus 8mm, and the height of the screen minus 32 mm.)

What are the dimensions (width and height) that allow the maximum screen area?

Your answer:

The dimensions for the most phones have a height that is about twice the width. Is that the case for your answer?

Make a selection:

1. No the height to width ratio is closer to 1 to 1
2. Yes, the height to width ratio is basically 2 to 1
3. No, the height to width ratio is closer to 3 to 1
4. No the height to width ratio is closer to 4 to 1

- *Saving money*

It is desired to build a pipeline from an at sea oil well to refinery on the shore. The oil well is 2 miles offshore and the refinery is 3 miles along the coastline. Building a pipe costs 500, 000*permileunderwaterand*300,000 per mile under land. Find the cost of the cheapest possible pipeline. (Problem, with figure, borrowed from this example 5).

Enter a number: ____________________________

- *Will you be in the water?*

The Statue of Liberty stands 92 meters high, including the pedestal which is 46 meters high. How far from the base is it when the viewing angle, theta, is as large as possible? figure

Your answer:

CHAPTER 10

Integration with Julia

To get started, we load the MTH229 package:

```
using MTH229
```

Quick background

Read more about this material here: integration.

For the impatient, in many cases, the task of evaluating a definite integral is made easy by the fundamental theorem of calculus which says that for a continuous function f the following holds for any antiderivate, F, of f:

$$\int_a^b f(x)dx = F(b) - F(a).$$

That is the definite integral is found by evaluating a related function at the endpoints, a and b.

The SymPy package can compute many antiderivatives using a version of the Risch algorithm that works for elementary functions. SymPy's integrate function can be used to find an indefinite integral:

```
f(x) = x^2
integrate(f)
```

$$\frac{x^3}{3}$$

Or a definite integral by passing in values a and b:

```
integrate(f, 0, 1)                 # returns a "symbolic" number
```

$$\frac{1}{3}$$

However, this only works *if* there is a known antiderivative $F(x)$ - which is not always the case. If not, what to do?

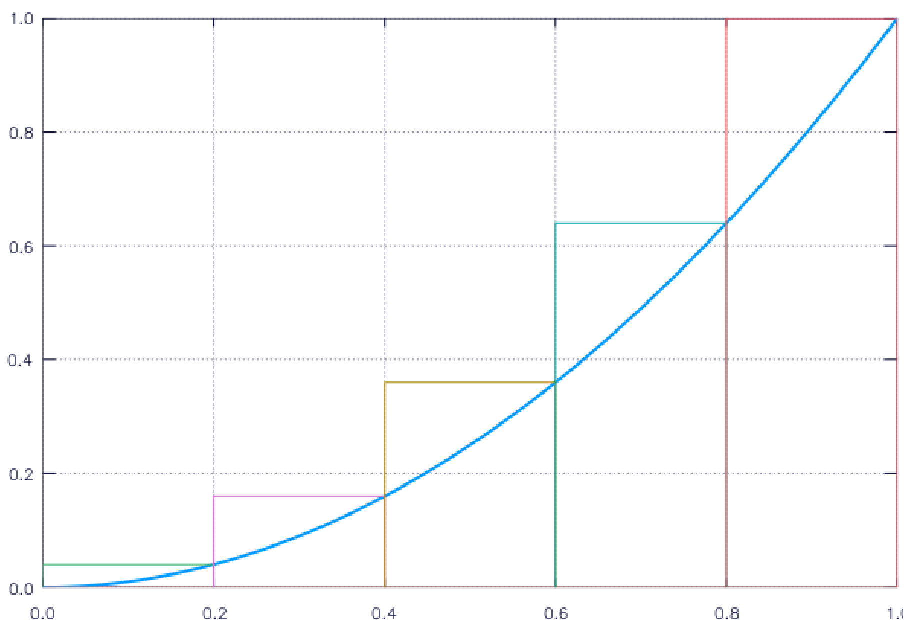
In this case, we can appeal to the *definition* of the definite integral. For continuous, non-negative $f(x)$, the definite integral is the area under the graph of f over the interval $[a, b]$. For possibly negative functions, the indefinite integral is found by the *signed* area under f. This area can be directly *approximated* using Riemann sums, or some other approximation scheme.

The Riemann approximation can help. The following pattern will compute a Riemann sum with equal-sized partitions using right-hand endpoints:

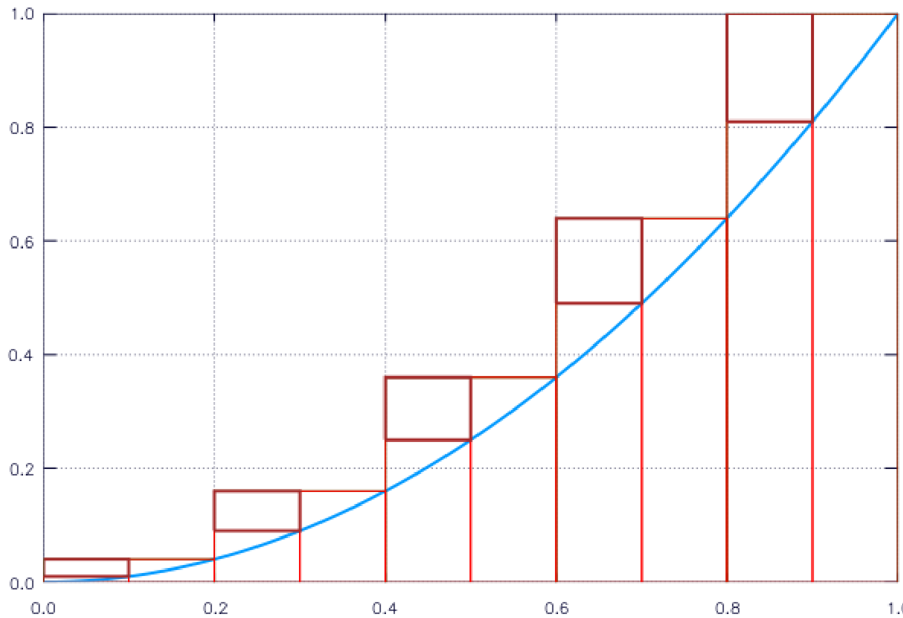
```
f(x) = x^2
a, b, n = 0, 1, 5                  # 5 partitions of [0,1] requested
delta = (b - a)/n                  # size of partition
xs = a + (1:n) * delta
fxs = [f(x) for x in xs]
sum(fxs * delta)                  # a new function `sum` to add up values in a contai
```

```
0.44000000000000006
```

That value isn't very close to $1/3$. But we only took $n = 5$ rectangles – clearly there will be some error, as we can see in this figure, where each approximating rectangle overestimates the area under the curve:



Bigger ns mean better approximations. This figure overlays the picture with $n = 10$ and emphasizes the excess area that is lost:



With large n the figures become too crowded to illustrate, but we can easily use a large value of n in our computations. With $n = 50,000$ we have the first 4 decimal points are accurate:

```
f(x) = x^2
a, b, n = 0, 1, 50_000                  # 50,000 partitions of [0,1] requested
delta = (b - a)/n
xs = a + (1:n) * delta
fxs = [f(x) for x in xs]
sum(fxs * delta)
```

```
0.3333433334
```

Note that only the first two lines of the calculuation needed changing to adjust to a new problem. As the pattern is similar, it is fairly easy to wrap the computations in a function for convenience. We borrow this more elaborate one from the notes (it is in the `MTH229` package) that works for some other methods beside the default right-Riemann sum:

```
function riemann(f::Function, a::Real, b::Real, n::Int; method="right")
  if method == "right"
     meth(f,l,r) = f(r) * (r-l)
  elseif method == "left"
     meth(f,l,r) = f(l) * (r-l)
  elseif method == "trapezoid"
     meth(f,l,r) = (1/2) * (f(l) + f(r)) * (r-l)
  elseif method == "simpsons"
```

```
    meth(f,l,r) = (1/6) * (f(l) + 4*(f((l+r)/2)) + f(r)) * (r-l)
  end

  xs = a + (0:n) * (b-a)/n
  as = [meth(f, l, r) for (l,r) in zip(xs[1:end-1], xs[2:end])]
  sum(as)
end
```

```
riemann (generic function with 1 method)
```

The Riemann sum is very slow to converge here. There are faster algorithms both mathematically and computationally. We will discuss two: the trapezoid rule, which replaces rectangles with trapezoids; and Simpson's rule which is a quadratic approximation. Each is invoked by passing a value to the `method` argument:

```
f(x) = x^2
riemann(f, 0, 1, 1000, method="trapezoid")
```

```
0.33333350000000006
```

And for Simpson's method:

```
riemann(f, 0, 1, 1000, method="simpsons")
```

```
0.3333333333333337
```

Base `julia` provides the `quadgk` function which uses a different approach altogether. It is used quite easily:

```
f(x) = x^2
ans, err = quadgk(f, 0, 1)
```

```
(0.3333333333333333,5.551115123125783e-17)
```

The `quadgk` function returns two values, an answer and an estimated maximum possible error. The ans is the first number, clearly it is 1/3, and the estimated maximum error is the second. In this case it is small (10^{-17}) – basically 0.

In summary we have these functions to find approximations to definite integrals when the fundamental theorem of calculus can't be employed:

- `integrate` - symbolically find a definite integral using the fundamental theorem of calculus, if possible.
- `riemann` - approximate a definite integral using either Riemann sums or a related method.
- `quadgk` - use Gauss quadrature approach to efficiently find approximations to definite integrals.

Questions

- Let $g(x) = x^4 + 10x^2 - 60x + 71$. Find the integral $\int_0^1 g(x)dx$ by hand by finding an antiderivative and then using the fundamental theorem of calculus.

Your answer:

- For $f(x) = x/\sqrt{g(x)}$ (for $g(x)$ from the last problem) estimate the following using 1000 Riemann sums:

$$\int_0^1 f(x)dx$$

Enter a number: ____________________________

- Let $f(x) = \sin(\pi x^2)$. Estimate $\int_0^1 f(x)dx$ using 20 right-Riemann sums

Enter a number: ____________________________

- For the same $f(x)$, compare your estimate with 20 Riemann sums to that with 20,000 Riemann sums. How many digits after the decimal point do they agree?

Make a selection:

1. They differ at the first place after the decimal point
2. They differ at the second place after the decimal point
3. They differ at the third place after the decimal point
4. They differ at the fourth place after the decimal point

5. They differ at the fifth place after the decimal point
6. They differ at the sixth place after the decimal point

Left Riemann

The left Riemann sum uses left-hand endpoints, not right-hand ones.

- For $f(x) = e^x$ use the left Riemann sum with $n = 10,000$ to estimate $\int_0^1 f(x)dx$.

Enter a number: __________________________

- The left and right Riemann sums for an increasing function are also lower and upper bounds for the answer. Find the difference between the left and right Riemann sum for $\int_0^1 e^x dx$ when $n = 10,000$. (Use your last answer.) What is the approximate value of the difference 1/100, 1/1000, 1/10000, or 1/100000?

Make a selection:

1. 1/100
2. 1/1000
3. 1/10000
4. 1/100000

Trapezoid, Simpson's

- The answer to $\int_0^1 e^x dx$ is simply $e^1 - e^0 = e - 1$. Compare the error (in absolute value) of the trapezoid method when $n = 10,000$.

Make a selection:

1. The error is about 1e-6
2. The error is about 1e-7
3. The error is about 1e-8
4. The error is about 1e-9
5. The error is about 1e-10

6. The error is about 1e-11

7. The error is about 1e-12

- The answer to $\int_0^1 e^x dx$ is simply $e^1 - e^0 = e - 1$. Compare the error of the Simpson's method when $n = 10,000$.

Make a selection:

1. The error is about 1e-9

2. The error is about 1e-11

3. The error is about 1e-13

4. The error is about 1e-15

(The error for Riemann sums is "like" $1/n$, the error for trapezoid sums is like $1/n^2$, and for Simpson's rule the error is like $1/n^4$.)

quadgk

- Use `quadgk` to find $\int_{-3}^{3}(1 + x^2)^{-1}dx$. What is the answer? What is the estimated maximum error?

The answer is:

Enter a number: ____________________________

The error is about

Make a selection:

1. The error is about 1e-6

2. The error is about 1e-8

3. The error is about 1e-10

4. The error is about 1e-12

5. The error is about 1e-14

- Use `quadgk` to find the integral of $e^{-|x|}$ over $[-1, 1]$.

Enter a number: ____________________

- The integral of $\sqrt{1-x^4}$ over $[-1,1]$ can not be found with the Fundamental Theorem of Calculus using an elementary function for an antiderivative. What is the *approximate* value?

Enter a number: ____________________

- The integral of $f(x) = \log(\log(x))$ over $[e, e^2]$ can not be found with the Fundamental Theorem of Calculus using an elementary function for an antiderivative.

The graph of $f(x)$ over the interval $[e, e^2]$ makes clear that the triangle formed by the line connecting $(e, f(e))$ and $(e^2, f(e^2))$, the x axis, and the line $x = f(e^2)$ will form a lower bound for the area under f. What is the area of this approximation?

Enter a number: ____________________

Use `quadgk` to find a much better estimate for this integral:

Enter a number: ____________________

Different interpretations of other integrals

The integral can represent other quantities besides the area under a curve.

A formula to compute the length of a the graph of the function $f(x)$ from a to b is given by the formula:

$$\int_a^b \sqrt{1+f'(x)^2}dx$$

For example, the arc-length of the square root function between $[0,4]$ is given by:

```
f(x) = sqrt(x)
l(x) = sqrt(1 + f'(x)^2)
ans, err = quadgk(l, 0, 4)
```

```
(4.6467837188706085,6.719390994752007e-8)
```

- Use the arc-length formula when $f(x) = \sin(x)$ and the interval is $[0, \pi]$. What is the answer for the length of a half period of the sine curve?

Enter a number: ____________________

- What is the arc length of the graph of the function $f(x) = x^x$ over $(0, 2)$?

Enter a number: ____________________________

Applications

We discuss an application of the integral to finding the volumes - not just areas.

A *solid of revolution* is a figure with rotational symmetry around some axis, such as a soda can, a snow cone, a red solo cup, and other common objects. A formula for the volume of an object with rotational symmetry can be written in terms of an integral based on a function, $r(h)$, which specifies the radius for various values of h.

If the radius as a function of height is given by $r(h)$, the the volume is $\int_a^b \pi r(h)^2 dh$.

So for example, a baseball has a overall diameter of $2 \cdot 37$mm, but if we place the center at the origin, its rotational radius is given by $r(h) = (37^2 - h^2)^{1/2}$ for $-37 \leq h \leq 37$. The volume in mm^3 is given by:

```
r(h) = (37^2 - h^2)^(1/2)
v(h) = pi * r(h)^2
ans, err = quadgk(v, -37, 37)
```

```
(212174.79024304505,2.9103830456733704e-11)
```

The volume in cubic inches, then is:

```
ans / (2.54 * 10)^3
```

```
12.947700103145083
```

Glass half full

- A glass is formed as a volume of revolution with radius as a function of height given the equation $r(h) = 2 + (h/20)^4$. The volume as a function of height b is given by $V(b) = \int_0^b \pi r(h)^2 dh$. Find $V(25)$. Show your work.

  Your answer:

- By trial-and-error or some more sophisticate approach, gind a value of b so that $V(b) = 455$. You only need to be accurate to the nearest integer.

Enter a number: ____________________________

- Now find a value of b for which $V(b) = 455/2$. This height will have half the volume as the height just found.

Enter a number: ____________________________

Compare the two values. Is the ratio of smallest to largest 1/2, more than 1/2 or less?

Make a selection:

1. The height to fill to 455/2 is exactly half the height to fill to 455
2. The height to fill to 455/2 is more than half the height to fill to 455
3. The height to fill to 455/2 is less than half the height to fill to 455