The College of Staten Island

Department of Mathematics

MTH 229

Calculus I Computer Lab

http://www.math.csi.cuny.edu/matlab/

MATLAB PROJECTS

STUDENT: _____

SECTION: _____

INSTRUCTOR: _____

BASIC FUNCTIONS

Elementary Mathematical functions		
MATLAB notation	Mathematical notation	Meaning of the operation
`sqrt(x)`	$\sqrt{x}$	square root
`abs(x)`	$|x|$	absolute value
`sign(x)`		sign of x ($+1$, -1, or 0)
`exp(x)`	e^x	exponential function
`log(x)`	$\ln x$	natural logarithm
`log10(x)`	$\log_{10} x$	logarithm base 10
`sin(x)`	$\sin x$	sine
`cos(x)`	$\cos x$	cosine
`tan(x)`	$\tan x$	tangent
`asin(x)`	$\sin^{-1} x$	inverse sine
`acos(x)`	$\cos^{-1} x$	inverse cosine
`atan(x)`	$\tan^{-1} x$	inverse tangent

The College of Staten Island
Department of Mathematics

MTH 229 Calculus Computer Laboratory
Course Outline

The main objective of this course is to reinforce calculus concepts and explore the application of calculus to solving problems by making use of a series of computer projects. The student will be first introduced to mathematical software. In particular, MATLAB software will be used in this course. MATLAB has capabilities for both numerical and symbolic calculations. It can also create graphical output so that the results can be visualized more readily.

The following projects are integrated with the material covered in courses MTH 230 *Calculus I with Pre-Calculus* and MTH231 *Analytical Geometry and Calculus I*. Therefore, full appreciation of these projects requires a solid understanding of the course material.

1. Using MATLAB as a Calculator

2. Plotting Graphs in MATLAB

3. More on Graphing with MATLAB

4. Graphical Solutions to Equations

5. Investigating Limits in MATLAB

6. Approximate First and Second Derivatives

7. Critical and Inflection Points

8. Newtons' Method

9. Optimization

10. Definite Integrals and Riemann Sums

Examinations: There will be a midterm and a final examination.

Optional Materials:

<u>Software</u>: MATLAB is installed in several of the campus computer labs. However, if you wish to work from home, you can purchase the MATLAB & Simulink Student Version R2007a at the student bookstore or online at *http://www.mathworks.com/academia*
Online documentation can be found at *http://www.mathworks.com/academia/student_center*

MTH229

The College of Staten Island
Department of Mathematics

Using MATLAB as a Calculator

1   Introduction

This project is designed to give you a brief introduction to the MATLAB software which will be used to help carry out your computer projects during this semester. This software is especially designed for mathematical, scientific and engineering applications.

Many of you probably have already used a scientific or graphics calculator like the TI-84. In its simplest application, MATLAB can be used just like a graphing calculator. In this project we learn how to turn "math into MATLAB." That is, we learn how to ask MATLAB math questions such as "what is 2+2?"

1.1   New MATLAB topics

   a.   Arithmetic Operations and Precedence Rules

   b.   Function Evaluation

   c.   Command Line Editor (correcting errors)

   d.   Arrays and Tables

   e.   Printing Text

   f.   Resizing and Moving Windows

1.2   New MATLAB commands

   a.   `;` – The semicolon suppresses the output.

   b.   `x=a:h:b` – This construct creates sequences of numbers.

   c.   `[x;y]'` – This construct creates a nice tabular look to lists of numbers.

1.3   On-line resources for MATLAB and the MATLAB projects

This project, and others, are available in an on-line format. Go to the following url for more information:

   http://www.math.csi.cuny.edu/matlab
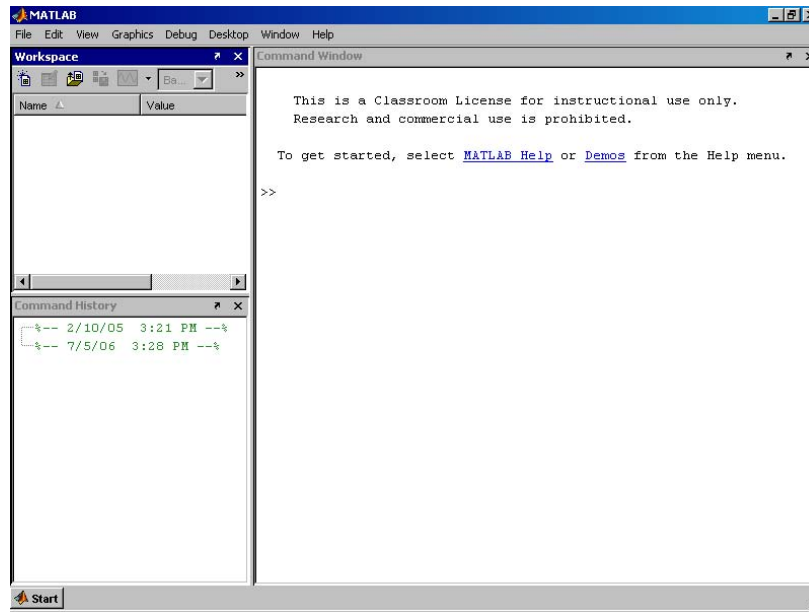
1.4   Starting MATLAB

Figure 1: Initial MATLAB window contains Command Window, Workspace viewer, and Command History panes.

MATLAB is started by double clicking its desktop icon (on a typical installation of MATLAB on Windows, other installations may vary). Figure 1 shows a typical startup window. It consists of several panes. Additionally other windows, such as a help window or plot window, may appear.

The visible panes in the figure are the **Command Window** which appears on the right side of the window. This window contains the command line where we type commands for MATLAB to interpret. This will be our main method to interact with MATLAB. Additionally, in the upper left pane is a Workspace viewer which shows the variables that are present in the MATLAB session, and in the lower left pane the Command History window displays the commands you have entered into the command window.

Commands are typed after the **prompt**: **>>**. After they are typed in, the **Enter** key is hit to send the command to the MATLAB interpreter. MATLAB either shows the answer or will respond with an error message indicating why it couldn't do what was requested.

Multiple commands may be entered on the same line if the commands are separated by a semicolon **;**, or a comma **,**. As well, a semicolon at the end of a command causes its output not to be displayed.

2 Arithmetic Operations

We begin our explorations with MATLAB by learning how to do simple **arithmetic operations**:

For instance, adding $2 + 2$ is done with

```
>> 2 + 2

ans =

    4
```

After typing in the command (2 + 2), we typed the Enter key. MATLAB responded with an answer of 4.

Some other examples are shown below:

	symbol	example	MATLAB response
addition	+	>> 3+4	ans = 7
subtraction	–	>> 5 – 9	ans = –4
multiplication	*	>> 5*9	ans = 45
division	/	>> 8/9	ans = 0.8889
exponentiation	^	>> 5^3	ans = 125

The only symbol that may need learning is the ^ for powers.

2.1 Assignment

When the command 2 + 2 is typed above, MATLAB evaluates it and responds with an answer of 4. It then forgets about this calculation because we didn't ask for it to be saved. In order to save values we can **assign** the output to a variable. After doing this, we can refer to the output by the variable name.

In the example below, we assign the value 18 to the variable a, the value 21 to the variable b, and the value $18 - 21 = -3 = a - b$ to the variable c.

```
command                 MATLAB response
>> a = 18                 a = 18
>> b = 21                 b = 21
>> c = a - b              c = -3
```

Once a value is assigned to a variable, MATLAB remembers it until that variable is reassigned.

Try predicting the responses to the following MATLAB commands to check your understanding.

```
                        MATLAB response
>> a = 3; b = 4;c = 5;
>> a=b*c
```

```
>> b/a
>> a=a-18
>> a^b
```

When you make an assignment, the variable name appears in the Workspace viewer which appears in the upper left pane of the initial MATLAB window. You can view the contents of the variable by double clicking it in the workspace viewer. Otherwise, you can enter just the variable name in the command line and its contents will be displayed.

Exercise 1:

   a. What is the output of the following commands:

```
>> a=3; b=4; c=5;
>> a + b/c
```

   (1) Circle one:
   1. 3 4/5
   2. 1.6
   3. 3.8
   4. 7/5

Example 1:
Assigning variables can simplify matters when used wisely. For example, when evaluating

$$\frac{(2-3)-(-3)}{(-1)+2}.$$

One way is to write out the whole expression at once:

```
>> ((2-3)-(-3))/((-1)+2)
ans = 2
```

There are many parentheses that are needed to get this right. This can make finding errors in our work tough. If we use spaces (which are ignored) and intermediate names, we can reduce the chance of a typing error:

```
>> top = (2-3) - (-3);
>> bottom = (-1) + 2;
>> top/bottom
ans = 2
```

This technique makes errors much easier to find.

Exercise 2:

a. Use assignment to help you compute

$$3 - \frac{3^2 - 2 \cdot 3}{2 \cdot 3 - 2}.$$

(2) Answer: _____

3   Order of Operations

What is

$$5 - 2/6?$$

Is it $5 - 1/3$ or 3 divided by 6? That is, $5 - (2/6)$ or $(5 - 2)/6$? We should know that the first is true because division happens before subtraction. What makes knowing this necessary is because there are two operations above, subtraction and division, and the answer will depend on which is done first. The order of operations act like a traffic cop directing the flow. MATLAB uses a fairly standard order of priority (precedence) for operations: addition and subtraction have the same priority, which is below multiplication and division. Powers have the highest priority. So typing the command

```
>> 5 - 2/6
```

returns an answer of 4.6667 or $5 - 1/3$, as division is done before the subtraction.
    In the following exercise you will review the basic order of operations.

3.1   Multiplication/Division vs. Addition/Subtraction

Exercise 3:
One of these things is **not** like the others. Which of these MATLAB commands is not like the other two? To help you out, try doing this with some values for a, b and c like

```
>> a= 3; b=13; c= 23;
```

a. **(3) Circle one:**
   1. `>> a - b * c`
   2. `>> a - (b * c)`
   3. `>> (a - b) * c`

b. **(4) Circle one:**
    1. >> a * (b - c)
    2. >> (a * b) - c
    3. >> a * b - c

c. **(5) Circle one:**
    1. >> a / b + c
    2. >> a / (b + c)
    3. >> (a / b) + c

d. **(6) Circle one:**
    1. >> (a + b) / c
    2. >> a + (b / c)
    3. >> a + b / c

e. Which operations have higher precedence?
    (7) Circle one:
    1. multiplication/division
    2. addition/subtraction

3.2 Multiplication/Division vs. Exponentiation

Exercise 4:
Repeat the same exercise with the following expressions.

a. **(8) Circle one:**
    1. >> a ^ (b * c)
    2. >> (a ^ b) * c
    3. >> a ^ b * c

b. **(9) Circle one:**
    1. >> a * (b ^ c)
    2. >> a * b ^ c
    3. >> (a * b) ^ c

c. **(10) Circle one:**
    1. >> a / b ^ c
    2. >> (a / b) ^ c
    3. >> a / (b ^ c)

d. **(11) Circle one:**
    1. `>> a ^ b / c`
    2. `>> (a ^ b) / c`
    3. `>> a ^ (b / c)`

e. Which operations have higher precedence?
    (12) Circle one:
    1. multiplication/division
    2. exponentiation

f. Using the rules you found, evaluate the following expression and rewrite it with parentheses showing the order in which MATLAB evaluates the operations. (For example, `a + b + c = (a + b) + c` and not `a + (b + c)`.) In this example there are 4 operations, so your answer should have 3 pairs of parentheses.

```
>>    7 - 3 ^ 2 / 9 + 4
```

(13) Answer:

3.3 Operations with the same precedence

How does MATLAB interpret the following commands?

```
>> 3 - 3 - 3
>> 6 / 3 / 2
>> 2 ^ 3 ^ 2
```

There is an ambiguity as the answers are different if the leftmost operation is done first compared to if the rightmost one is.

Exercise 5:
Evaluate the following expressions first using *left-to-right* order, and then from *right-to-left*. Enter both answers separated by a comma.

   a. $3 - 3 - 3$
      (14) Answer: _____

   b. $6/3/2$
      (15) Answer: _____

   c. `2^3^2`
      (16) Answer: _____

Exercise 6:
Investigate the expressions below to see the order in which the operations are performed by MATLAB. (For example, does $5 - 3 - 2 = (5 - 3) - 2$ or $5 - (3 - 2)$?)

```
>> 5 - 7 - 8
>> c - a + b - c
>> 5 / 7 / 8
>> 5 / 7 * 8 * 9
```

What rule(s) does MATLAB use when evaluating expressions with two or more operations of the same priority?
(17) Circle one:
1. from right to left
2. from middle to end
3. from left to right

Exercise 7:
Practice what you have just reviewed to evaluate the following. Let

```
>> a=4;b=5;c=8;
```

   a.

$$\frac{a^b - c/b}{c - a}$$

      (18) Answer: _____

b.

$$\frac{a^{(c-b)}}{c-b}$$

(19) Answer: _____

c.

$$\frac{a^{3/2}}{b}$$

(20) Answer: _____

d.

$$\frac{a-b(c-a)}{c-a}$$

(21) Answer: _____

Note: A full explanation of MATLAB's priority rules is attached to this project in the **Reference Section**. You may want to review this to check your answers.

3.3.1 Reusing previous commands

You can save some typing if you learn how to reuse and edit your previously entered commands. The **Command History** window shows the commands you've typed during a MATLAB session. Double clicking on a command will paste it into the command window. The command history may also be accessed inside the command line by using either the up or down arrow to scroll through your previous commands.

Once a command is at the >> prompt, you may make changes to it. Use the left or right arrows to move around, or the mouse. Then you may insert text or delete existing text.

Finally, you may type Enter to execute the new command. You do not need to have the cursor at the end of the line to do this.

4 Function evaluation

MATLAB allows much more than just the basic arithmetic operations. Extra functionality is provided by functions, such as sin, cos, or sqrt. A function is referred to by its name and used by calling the function with its argument(s) enclosed in matching parentheses. A list of basic functions and their MATLAB equivalents is attached to the end of the project in the Reference Section.

For instance, the value of $\sqrt{15}$ is given by

```
>> sqrt(15)                    #  ans = 3.8730
```

(The text after and including the number sign symbol is a comment and will be ignored if typed in.)

Just typing the function name, without the parentheses, will show its definition.

Here are a few more examples of function evaluations. Note that to get $\pi = 3.1415\ldots$, you type pi; to get the number $e = e^1 = 2.7182\ldots$, you type exp(1).

When trying these examples, if you get an error message, check that you have spelled the function name correctly. For example, if you type sqt(3) (you misspelled sqrt), MATLAB responds with the error message

```
??? Undefined function or variable sqt.
```

Try evaluating the following using MATLAB:

```
                          MATLAB response
>> sqrt(216)              14.6969
>> pi                     pi = 3.1416... is built-in
>> exp(1)                 ans = 2.7183    e is not built-in
>> sin(pi/4)              ans = 0.70711
>> x = pi/3               x = 1.0472    assigns x
>> tan(x)                 ans = 1.7321, as x = pi/3 by above
>> sin(x)^2 + cos(x)^2    ans = 1, again, as x = pi/3
```

Note: you will get better precision if you do not round off intermediate computations. Try typing the following to see an example:

```
>> pi/4
>> sin(0.785)
>> sin(pi/4)
>> sqrt(2)/2
```

When trigonometric functions are evaluated in MATLAB, arguments must be specified in radian measure—<u>not</u> in degrees. Recall, to convert from degrees to radian you multiply by $\pi/180$.

Exercise 8:
Use MATLAB to evaluate the following expressions.

a. Calculate the sine of 40 degrees using MATLAB. MATLAB uses radians for all angle measurements. You will need to convert degrees to radians first.
    (26) Answer: _____

b. Evaluate $\sin^2 65°$
    (27) Answer: _____

c. Evaluate $e^{(10-8.5)/3}$
    (28) Answer: _____

d. Evaluate $\arcsin(\sin(3\pi/4))$
(29) Circle one:
1. $3\pi/4$
2. $-5\pi/4$
3. $\pi/4$

5   Vectors

We will often want to apply a function to many different values of x. We'd like to do this in the most convenient manner. For example, to compute the function $f(x) = x^2 \cos^3(x)$ for $x = \pi/3, \pi/4$ and $\pi/6$ we can save some work by assigning a value to x and then computing:

```
>> x=pi/3; x^2 * cos(x)^3
>> x=pi/4; x^2 * cos(x)^3
>> x=pi/6; x^2 * cos(x)^3
```

This allows us to make a single change to x per line, instead of changing it in both places.

Although the above technique is useful, there are better ways to do this task, as the MATLAB language is written to naturally apply the same function to many different values at once. In order to do so we need to learn two things:

a. How to store more than one number into a variable (vectors)

b. How to apply a function to all the values of the vector simultaneously.

5.1   Defining vectors

We use the term vector to describe a MATLAB variable that contains lots of numbers at once. Vectors are made in MATLAB using the square brackets `[]`. (MATLAB refers to vectors as arrays, a more general concept.)

The simplest way to make a vector is to just type in the numbers you want inside of matching `[]`:

(Don't type the part with the `%` symbol. This is a comment to you.)

```
>> x = [1,1,2,3,5,8,13,21]      % start of Fibonacci sequence
>> x = [1 1 2 3 5 8 13 21]      % commas are optional
>> somePrimes = [2,3,5,7,11,13,17,19,23]
```

Exercise 9:

Store the number $1, 2, 3$ in a vector named x. Answer the following for this vector.

    a. What is `x+x`?
        (30) Circle one:
        1. The vector `[1, 4, 9]`
        2. The vector `[2, 4, 6]`
        3. An error

    b. What is the output of `x * x`
        (31) Circle one:
        1. The vector `[1, 4, 9]`
        2. The vector `[2, 4, 6]`
        3. An error

5.2   Arithmetic sequences

Many of the vectors of numbers we will deal with will be arithmetic sequences:

$$a, a + h, a + 2h, a + 3h, \ldots, a + kh = b, \qquad h > 0.$$

We think of this in two ways:

    a. Numbers between a and b separated by a step size of h

    b. A certain number $(k + 1)$ of numbers evenly spaced between a and b.

There are two different ways in MATLAB to generate such sequences, depending on how you think of the values in the sequence.

5.2.1   numbers separated by a step size h

To generate a sequence of numbers separated by 1 is done using the `:` symbol, as in `a:b`:

```
>> 1:5                          % 1 2 3 4 5
>> -1:5                         % -1 0 1 2 3 4 5
>> -(1:5)                       % -1 -2 -3 -4 -5
```

The last example shows that the minus sign here means multiply each entry by -1.
    If we want a step size of h we use this syntax: `a:h:b`. For instance:

```
>> evens = 0:2:10               % even numbers 0,2,4 ... 10
>> evens = 0:2:9                % even numbers 0,2,4 ... 8. Stops at 8
>> skip15 = 0:15:60             % 0,15,30,45,60 -- class is almost done
>> skip15 = 15*(0:4)            % same thing!
>> skipafew = 1:98              % 1 2 skipafew 97 98
```

Exercise 10:
Find MATLAB commands which generate the following lists. Make sure your answer is correct.

   a. The odd numbers 1,3,...99
      (32) Answer:

   b. The numbers 10,20,30,...120
      (33) Answer:

5.2.2 A fixed number of numbers

When plotting functions we will desire a lot (say 100 or a 1000) of evenly-spaced numbers between two points. Rather than figure out the step size between the points it is more convenient to specify how many linearly spaced points we want. This is done with the `linspace` function. Using it as `linspace(a,b)` will produce 100 numbers between a and b. You can override the default of 100 using a third argument:

```
>> linspace(0,5,6)          % ans = [0 1 2 3 4 5] (6 numbers)
>> linspace(0,pi,3)         % ans = [0, pi/2, pi] (3 numbers)
```

Exercise 11:

   a. What `linspace` command produces this output:

      1.000 1.500 2.000 2.500 3.000 3.500 4.000

      (34) Circle one:
      1. `linspace(1,7,4)`
      2. `linspace(1,4,1/2)`
      3. `linspace(1,4,7)`
      4. `linspace(1,1/2,4)`

   b. What is the last value output by the command

      `>> linspace(0,pi)`

      (35) Answer: _____

5.3 Applying arithmetic operations to vectors

In MATLAB the basic object is a matrix (a rectangular collection of numbers). As such, the default definitions for +, -, ∗, /, and ˆ are the matrix definitions. What we will want is a little different. This means we will need to be careful when we multiply, divide or take powers of vectors.

This will be discussed more in the next project. For this project, we focus on arithmetic operations which work as we would like: addition of vectors of the same size, and multiplication of a vector times a number (a scalar).

For example

```
>> x = 1:5              % the numbers 1 2 3 4 5
>> x + x               % the numbers 2 4 6 8 10
>> 3*x                 % the numbers 3 6 9 12 15
>> 3*(x - 3)           % subtract 3 then multiply:
                       %    -6 -3 0 3 6
```

This allows us to do simple transformations, such as the conversion from Celsius to Fahrenheit, or back, given by these formulas:

$$F = 9/5 C + 32 \quad \text{or} \quad C = 5/9(F - 32)$$

This is the basis of the last exercise.

Exercise 12:
Two common measures of temperature are the Fahrenheit and the Centigrade scales.

a. Find room temperature in Celsius (F = $68°$)
   (36) Answer: _____

b. Find the average body temperature in Celsius (F = $98.6°$)
   (37) Answer: _____

c. Let X be a vector of Fahrenheit values between -100 and 100 in step size of 20, and Y be the corresponding Celsius values. Which MATLAB commands give this?
   (38) Circle one:
   1. `X=-100:10:100; Y=9/5*X+32`
   2. `X=-100:20:100; Y=9/5*x+32`
   3. `X=-100:10:100; Y=5/9*(X-32)`
   4. `X=-100:20:100; Y=5/9*(X-32)`

d. If two vectors are the same length, a table can be made from them that allows you to compare their entries. The syntax is either `[X;Y]` or `[X;Y]'`. Look carefully at the table of X and Y values. At what temperature is the Celsius and Fahrenheit measurement the same?
   (39) Answer: _____

---

6   Reference Section

6.1   Basic MATLAB commands

Description	Symbol(s)	Comments
Number (scalar) assignment	`a=7.25;    b=1/12`	
Scalar arithmetic	`a+b; a-b; a*b` `a/b; a^b`	
Vector assignment	`x=a:h:b` `x=[2  0  4  -7]`	Fills array x with $a, a+h, a+2h, \ldots, b$ Assigns sequence $2, 0, 4, -7$ to x
Scalar/vector operations	`a*x, a+x` `a-x, x/a`	If `x = [3 0 -1]` then `2*x = [6 0 -2]` `x-2 = [1 -2 -3]`, etc. Careful: `a/x` and `x^a` won't work.
Pointwise vector operations	`a./x, x.^a` `x.*y, x./y`	If `x = [3 0 -1]`, `y=[8 2 2]`, then `x.^2=[9 0 1]`, `x.*y=[24 0 -2]`, etc. The dot (`.`) makes it pointwise.
Separator	`;` (semicolon)	separates commands; suppresses output separates rows in a table or matrix
Table	`[x;y]` or `[x;y]'`	Two-row or two-column table

6.2   Order of Operations

MATLAB uses the following symbols for arithmetic operations:

Operation	Symbol	Precedence	Comment
Exponentiation	`^`	3	Highest precedence
Multiplication	`*`	2	
Division	`/`	2	
Addition	`+`	1	
Subtraction	`-`	1	Lowest precedence

If an arithmetic expression contains nested parentheses, then the expressions contained within the innermost parentheses are evaluated first. In the absence of parentheses, the **precedence rules** decide the order of evaluation. The following rules apply:

1. All operations with a higher precedence are carried out before those of lower precedences. Thus exponentiation is carried out first, then comes multiplication and division, and finally, addition and subtraction.

2. If two operations have the same precedence then the operation on the left is carried out first. This is called the left-to-right scan rule.

The examples in the following table show how the precedence rules help interpret arithmetic expressions based on these rules. Try them out on the computer and make sure that you understand how each expression is interpreted.

Expression	MATLAB value	Interpreted as
2*3+4*5	26	(2*3)+(4*5)
2+3*4	14	2+(3*4)
(2+3)*4	20	parentheses decides the order
2/5*4	1.6	(2/5)*4
2/(5*4)	0.1	parentheses decides the order
2^3^2	64	(2^3)^2
2^(3^2)	512	parentheses decides the order
5/2/4	0.625	(5/2)/4
5/(2/4)	10	parentheses decides the order

6.3   Basic Functions

Elementary Mathematical functions		
MATLAB notation	Mathematical notation	Meaning of the operation
sqrt(x)	$\sqrt{x}$	square root
abs(x)	$|x|$	absolute value
sign(x)		sign of x (+1, −1, or 0)
exp(x)	e^x	exponential function
log(x)	$\ln x$	natural logarithm
log10(x)	$\log_{10} x$	logarithm base 10
sin(x)	$\sin x$	sine
cos(x)	$\cos x$	cosine
tan(x)	$\tan x$	tangent
asin(x)	$\sin^{-1} x$	inverse sine
acos(x)	$\cos^{-1} x$	inverse cosine
atan(x)	$\tan^{-1} x$	inverse tangent

NOTE: MATLAB may give unexpected results when working with negative numbers, since its default is to assume inputs and outputs are complex numbers. Try evaluating sqrt(-4), (-8)^(1/3), and log(-5), to see what happens. Note that i represents $\sqrt{-1}$.

MTH229

The College of Staten Island
Department of Mathematics

Plotting Graphs in MATLAB

1   Introduction

Mathematically, a function f is a rule that assigns to each value x in its domain, a corresponding value of y in its range. The graph of a function is then the collection of all points (x, y) such that $y = f(x)$, sketched in the Cartesian plane. Of course we can't realistically expect to draw the (typically) infinite collection of points. In a calculus class we learn to sketch graphs by focusing on their important features: zeroes, asymptotes, limits, points of discontinuity or non-smoothness, relative maxima and minima, and points of inflection. With these important features understood, a sketch then can be drawn accurately where need be and filled in broadly otherwise.

With MATLAB we take a different approach. To plot the function f over the interval $[a, b]$, we actually plot as many pairs of points (x, y) as necessary to ensure an accurate representation of the graph. How many points? There are no good rules. We'll see examples (lines) where two are enough. As well, we will see examples where we can't possibly take enough points to do what we want. We'll just have to get used to experimenting to find the correct amount.

With this approach, we need to be able to do the following:

a.   generate the x values of the points in our graph,

b.   generate the corresponding y values,

c.   plot the points and connect them with lines.

We've seen how to make regularly spaced x values using either the `a:h:b` construction or the `linspace` function. In this project we'll learn how to make the corresponding y values, and how to then make the desired plot.

1.1   New MATLAB topics

- Vector mathematics and the dot symbol

- Graphing functions using pairs of vectors

- Exploring a graph with the command line and the mouse

- Annotating a graph

- Printing a graph

1.2   New MATLAB commands

- `a:h:b`, `linspace(a,b,n)` – used to generate regularly spaced sequences of points.

- The dot operators: `'.*'`, `'./'`, `'.^'`. – used in vector mathematics.

- `plot(x,y)` – plot the two lists of numbers

2   Graphing with MATLAB

Think back to how you first learned to graph a function or an equation. What would you do if you wanted to plot a graph of the parabola $y = x^2$ over the interval $-2 \leq x \leq 2$? We could choose a set of x values, say, $x = -2, -1, 0, 1, 2$, then square each x value to determine the corresponding y value ($y = 4, 1, 0, 1, 4$). These might be displayed together, as in the following table:

x	-2	-1	0	1	2
y	4	1	0	1	4

We would then mark each corresponding (x, y) pair as a point on a Cartesian coordinate system, and connect the points with straight lines.

Example 1:
To graph $f(x) = x^2$ over the interval $[-2, 2]$ using MATLAB we just need to know that the `plot` function will make the desired plot. Then we have (the `%` and beyond are comments and need not be typed in.)

```
>> x = [-2 -1 0 1 2]      % creates the vector x=[-2 -1 0 1 2]
>> y = [ 4  1 0 1 4]      % creates the y values
>>  [x;y]                 % display as a table
ans =
  -2  -1   0   1   2
   4   1   0   1   4
>> plot(x,y)              % plot points and lines
```
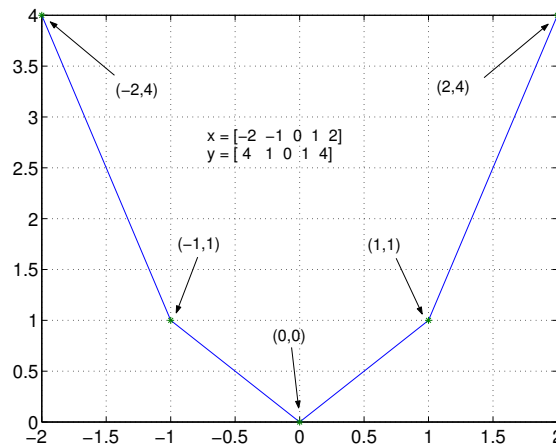
Figure 1: The function $f(x) = x^2$ plotted with 5 points.

Figure 1 shows that our graph isn't quite what we expected. Rather than looking like a familiar parabola, we see a sequence of straight lines. How can we fix that? By taking more points. We have an easy way of taking more points, we can generate a 100 of them with the command `linspace(-2,2)` and, if need be, 1,000 with the command `linspace(-2,2,1000)`. However, we don't want to do all the squaring by hand. Rather MATLAB should do the work. The next example shows how to do this for $f(x) = x^2$, and later on in this project we'll see how to do this for other functions.

```
>> x= linspace(-2, 2)
>> y=x.^2;                       % notice the ".^" and not just "^"
>> plot(x,y)                     % makes the plot
```
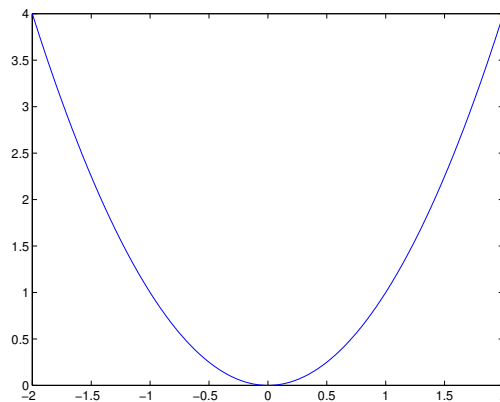

Figure 2: The function $f(x) = x^2$ plotted using 100 points.

Using more points requires no more labor on your part than using just 5 points. If we didn't have enough we'd take more and replot. However, our graph (in Figure 2) now looks okay. Even though it may no longer be evident, the graph still consists of a sequence of straight lines!

We repeat the previous example with a different function.

Example 2:
Graph the function $f(x) = e^x$ over the interval $[-1, 1]$.

We'll need to know that in MATLAB the function `exp(x)` performs e^x. Other than that, this example follows the three steps in plotting:

a. We want to plot the function over the interval $[-1, 1]$. To do so, we first choose evenly spaced points between -1 and 1 with a step size of 0.5

```
>> x = -1:0.5:1              % no ';' means you see all the values
```

b. We then define the y values for each x value using `exp`:

```
>> y = exp(x)               % again you see all the values as no ';'
```

c. Finally we use `plot( )` to view the function's graph:

```
>> plot(x,y)                % does your answer match the graph shown?
```
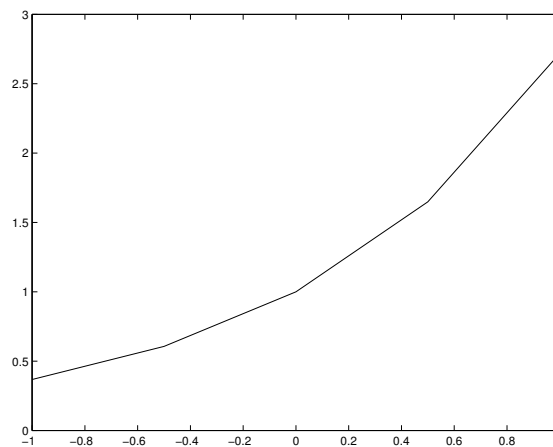

Figure 3: The function $f(x) = e^x$ plotted with 5 points.

As shown in Figure 3, this graph created with 5 sample points in $[-1, 1]$ is obviously not so smooth. To obtain a smooth looking curve one needs to define more x points. The exact number varies depending on how rapidly the function varies over its domain. We start with the simple default of 100 using `linspace` and see if this is enough:

```
>> x = linspace(-1,1)
>> y = exp(x)
>> plot(x,y)
>> grid                     % add a grid to the graph
```
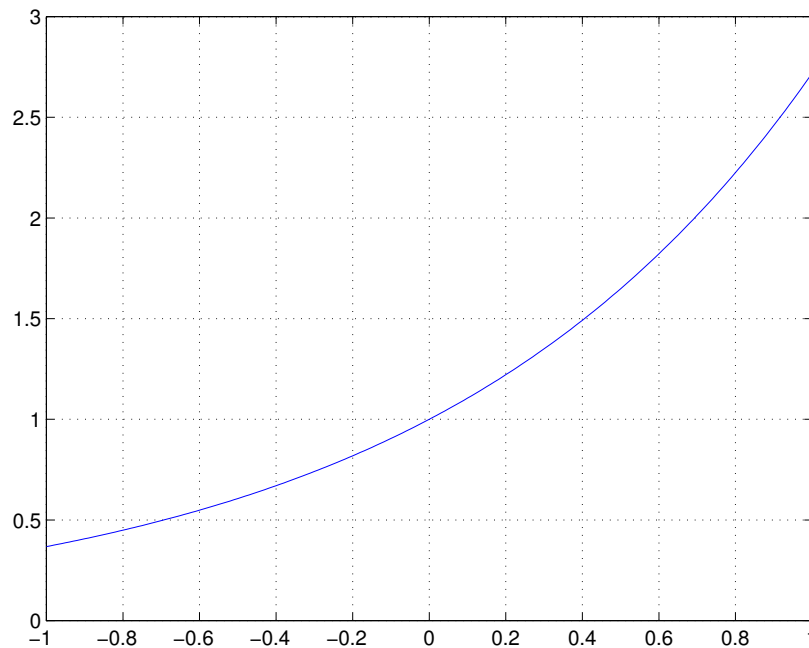
Figure 4: The function $f(x) = e^x$ plotted with 100 points.

The graph in Figure 4 shows that 100 points is sufficient to present a smooth looking curve.

It is important that even though only the line defining the values for x is different, you must again evaluate the second and third lines. The values stored in y do not update automatically. If you forgot to recreate the y values, you would be trying to pair off the 100 x values with only 5 y values and an error would occur. For such a simple function we can avoid this step by defining y within the plot function, as in

```
>> plot(x, exp(x))
```

Exercise 1:
Create a graph of $y = \cos 4x$ over $[0, \pi]$. To illustrate what happens when there are too few points in your domain, first try a step size of $\pi/10$ (pi/10).

a. Which command gives the desired values for x?
   (1) Circle one:
   1. x=0:pi/10:pi
   2. x=0:pi:pi/10
   3. x=linspace(0,pi)

   b. Which command gives the correct answer for y?
   (2) Circle one:
   1. `y = cos(4x)`
   2. `y = cos4*x`
   3. `y = cos(4*x)`

   c. Plot your graph with the `plot` command. You don't need to turn it in.

   d. Redo your plot, this time using the command `>>x=linspace(0,pi)` to define the x array. Which plot looks more like the plot of a cosine curve?
   (3) Circle one:
   1. The first one
   2. the second one
   3. both of them

Exercise 2:
We wish to plot the function $f(x) = e^{\cos(x)}$ over the interval $[0, 2\pi]$.

   a. What command generates a sufficient number of values for x?
   (4) Circle one:
   1. `linspace(0,2*pi)`
   2. `linspace(0,100,2*pi)`
   3. `0:2*pi`
   4. `0:2*pi:0.01`

   b. Which command will generate the corresponding y values:
   (5) Circle one:
   1. `exp^cos(x)`
   2. `e^cos(x)`
   3. `exp(cos(x))`
   4. `exp(x)cos(x)`

3   Algebraic expressions with vectors

We now know pretty well how to create the values for x using `linspace` or `a:h:b`. To create the values of y is a little harder. We want to create the values simultaneously and so must use the proper MATLAB syntax.

For concreteness we call a single number a scalar and a set of numbers a vector. (Although MATLAB thinks of them both as examples of arrays.)

Suppose we have two terms `a` and `b` and want to find one of these arithmetic operations: `a+b`, `a-b`, `a*b`, `a/b`, or `a^b`. The answer depends on the whether the terms are scalars or vectors:

If both `a` and `b` are scalars Then no special care is needed.

If one of `a` or `b` is a vector When one of the terms is a scalar and the other a vector then two cases won't work as expected. First with powers, `a ^ b`, involving a vector, You will need to use the notation `.^` in place of `^`. This is why we used `x.^2` in our first example, instead of simply `x^2`. Second, when dividing by a vector you need to use the notation `./` and not simple `/`. These extra "**dot**'s" are important to learn.

If both `a` and `b` are vectors When both terms are vectors **and both have the same length**, then `a*b`, `a/b`, **and** `a^b` need to use a "dot," as in

```
a .* b,    a ./ b,    a .^ b
```

(In fact, you could always use the dot form in the other instances, but it is not required, and looks really bad.)

Example 3:
Let's see what happens if you don't know how to work with arrays. We define x to be the numbers 1 through 5:

```
>> x = 1:5
x = 1   2   3   4   5
```

We might try to multiply x by 10.

```
>> 10*x
ans = 10    20    30    40    50
```

Which does what we expected. What about dividing by 10?

```
>> x/10
ans = 0.10000  0.20000  0.30000  0.40000  0.50000
```

What about adding 10?

```
>> x + 10
ans = 11   12   13   14   15
```

So what's the fuss? Well we've been lucky. Try to divide x into 10:

```
>> 10/x
ERROR, ERROR WILL ROGERS.
```

How about squaring x:

```
>> x^2
ERROR, ERROR WILL ROGERS. I'M BEGINNING TO SMOKE!
```

How about trying to multiply x by itself

```
>> x*x
ERROR, ERROR WILL ROGERS. I CAN'T TAKE IT ANYMORE. READ THE MANUAL.
```

Actually, the error messages say something like the first one:

```
??? Error using ==> /
Matrix dimensions must agree.
```

For $10/x$, MATLAB is trying to use matrix division which is not defined for this problem. We want our division to be element by element, so we need to specify the extra "dot."

Go back and check that

```
>> 10 ./ x
>> x .^ 2
>> x .* x
```

work as expected.

Here are some examples where the vector notation is used:

Example 4:
Plot $y = \sin x + \cos 3x$ over the domain $[0, 2\pi]$.

```
>> x = linspace(0,2*pi);
>> y = sin(x)+cos(3*x);
>> plot(x,y)
```

We didn't need any dots as 3*x is a scalar times an vector, The `sin` and `cos` functions are smart about vectors, and the + is between two vectors of the same size.

Example 5:
Plot $y = e^{-x/2} \cos 6x$ over the domain $[0, 10\pi]$:

```
>> x = linspace(0,10*pi);
>> y1 = exp(-x/2);         % no dot for -x/2 since 2 is a scalar
>> y2 = cos(6*x);          % Break up the computation into bite-sized pieces
>> y = y1.*y2;             % dot needed since y1 and y2 are both vectors
>> plot(x,y)
```

Again the **dot** . before the * means that multiplication of the two same-sized vectors y1 and y2 is to be carried out element-by-element. To minimize the chance of errors, we broke the problem into intermediate calculations by using two variables y1 and y2.

Example 6:
Plot $y = 1/(x^2 - 1)$ over the domain [2,5]:

```
>> x = 2 : 0.1 : 5;
>> y = 1./(x.^2-1);
>> plot(x,y)
```

Here the "dot" is used twice—powers (x^2) always get a dot when a vector is involved, and the division by a vector requires the extra dot.

Exercise 3:
Define a, b and c by

```
>> a = 1:2:20; b = 1:10; c = 1:2:10;
```

Which of the following is defined?

   a.   b+c
       (6) Circle one:
       1. yes **2.** no

   b. a + b
       (7) Circle one:
       1. yes **2.** no

   c. a./ b
       (8) Circle one:
       1. yes **2.** no

   d. a * b
       (9) Circle one:
       1. yes **2.** no

Exercise 4:

Let x=[1 2 3]. Translate the following math statements into MATLAB commands. To help, the value for the function when x=[1 2 3] is given in parentheses.

a. Write MATLAB commands to compute:

$$\cos(x)\sin(x)$$

```
ans =
  0.4546  -0.3784 -0.1397
```

(10)

b. Write MATLAB commands to compute:

$$\sin(x)^2$$

```
ans =
  0.7081  0.8268  0.0199
```

(11)

c. Write MATLAB commands to compute:

$$\sin(x^2)$$

```
ans =
   0.8415  -0.7568   0.4121
```

(12)

d. Write MATLAB commands to compute:

$$f(x) = 7x^2 \sin(\frac{1}{7x^2})$$

```
ans =
   0.9966   0.9998   1.0000
```

(13)

e. Write MATLAB commands to compute:

$$f(x) = x - \frac{\cos(x) - \sin(x)}{\sin(x) + \cos(x)}$$

```
ans =

   1.2180   4.6877   1.6675
```

(14)

f. Write MATLAB commands to compute:

$$f(x) = \frac{1}{10}(x - \frac{x^{3/2}}{10})^2$$

```
ans =

   0.0810   0.2949   0.6152
```

(15)

3.1   How to submit graphs

If you turn your work in on paper, you may be asked to print out a graph and attach it to your project. Printing a graph can be done using the printer icon or `Print...` dialog under the `File` menu of the figure window.

    If you submit your work through the web interface, you must attach your graph to your project when you save or submit your work. This is done with the following steps:

a. First save your graph as a JPEG file using the `Save As...` under the `File` dialog of the **figure window** (Figure 5). If you name your file with a `.jpg` extension, MATLAB will save the figure in this format. You should save the figure in your `My Documents` directory, your desktop, or some other place that is easy to find.

b. You then attach the saved jpeg image using the `browse` button in the web form. When the project is saved or submitted your image file(s) will also be included. You can verify what is saved by clicking the new link accompanying the question.

    You do not need to do this each time you save the project, the files are stored on the server. They can be overwritten if a new file is attached.
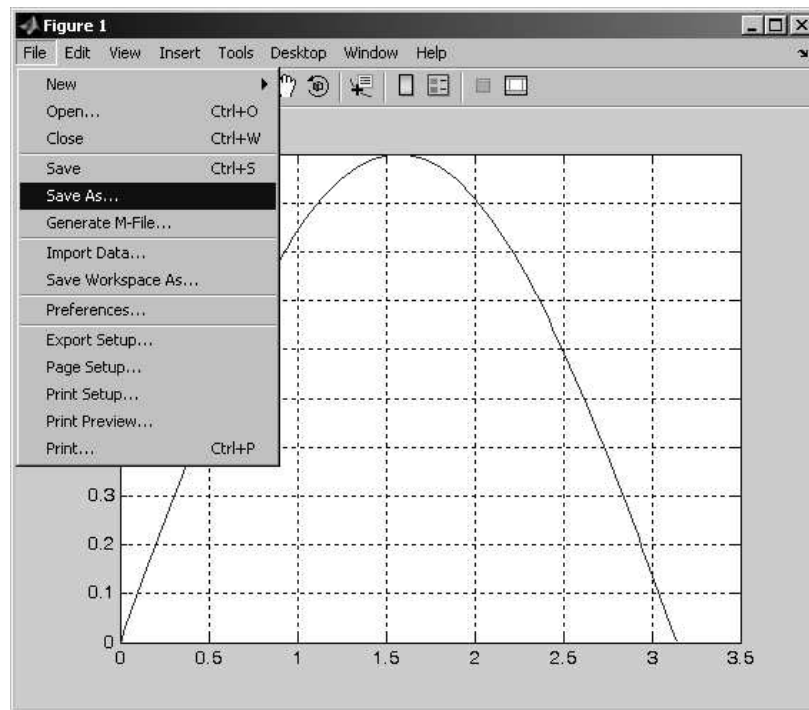

Figure 5: To save graphs to attach to project for web submission, use the `Save As...` dialog under the `File` menu.

Exercise 5:
Graph the function $f(x) = \sin((\pi/2)x) + \sin((2/5)\pi x)$ over the interval $[0, 40]$.

a. How many peaks (relative maxima) does the graph have?
   (16) Answer: _____

b. This function is periodic. How many periods are graphed in $[0, 40]$?
   (17) Circle one:
   1. 2
   2. 3
   3. 4
   4. 5
   5. none of the above

c. Estimate from your graph the value of $f(10)$ to at least 1 decimal point.
   (18) Answer: _____

d. Upload your graph.
   (19) **Attach your graph to the worksheet.**

Exercise 6:

a. Graph the function $f(x) = \cos^2(x) - \sin^2(x)$ over the interval $[-2\pi, 2\pi]$. Use 100 points in the domain.
   (20) **Attach your graph to the worksheet.**

b. Does the graph resemble any graph that you are familiar with?
   (21) Circle one:
   1. $\cos 2x$
   2. $\cos x/2$
   3. $\cos x$

Plotting Graphs in MATLAB"

Exercise 7:
For this exercise we look at the graph of the polynomial function $f(x) = x^3 - 20x^2 + 10x - 1$.

   a. First plot the function over the interval $[-10, 10]$. What is the approximate range for the y-axis?
      (22) circle one:
      1. [-10,10]
      2. (-10,10)
      3. [-3100,0]
      4. $[0, 2\pi]$

   b. We wish to investigate when (if) this function is positive. We can't readily tell from our graph so we will replot over a smaller domain. Which of these domains seems appropriate for this task?
      (23) circle one:
      1. [0,500]
      2. [0,10]
      3. [-1,1]
      4. $[0, 2\pi]$

   c. Replot the graph over the selected domain. Turn on the grid by entering the command

```
>> grid
```

      From your graph, which of these x values have $f(x) > 0$?
      (24) Circle all that apply:
      1. 0
      2. 0.25
      3. 0.50
      4. 0.75

MTH229

The College of Staten Island
Department of Mathematics

More on Graphing with MATLAB

1   Introduction

We've seen that basic plots may be produced in MATLAB by following three steps: produce an appropriate selection of x values; generate the corresponding $y = f(x)$ values; plot the two vectors with the command `plot(x,y)`.

   In this project we show how to handle various situations that arise when plotting functions that require us to think before we start making our plots. These include: figuring out an appropriate domain for the plot; plotting functions with vertical asymptotes; plotting more than one function on the same graph; and plotting functions which are defined differently over different intervals. Additionally, we learn how to add color, labels, and titles to a figure.

1.1   New MATLAB Topics

   a.   Plotting Several Functions on the Same Graph

   b.   Plotting Functions Defined by Several Equations

   c.   Exploring Graphs with the Mouse

   d.   Annotating Graphs

1.2   New MATLAB Commands

   a.   `zoom [on off]` – use the mouse to 'zoom' in on parts of the graph.

   b.   `plot(x1,y1,x2,y2)` – plot multiple graphs at once.

   c.   `hold [on off]` – hold the current graph.

2   Using `zoom` to explore a figure

Once a function is plotted, we typically want to use the figure to answer questions about the function. Example questions are: When does the function equal 0? When is the function at its lowest point? Where is the function increasing? These can be answered by focusing on one or two points on the graph. To read the values of these points we can use the label on the x and y axes. However, to get more precise answers it is useful to be able to "zoom" in on the graph to narrow the range of

the axes. This can be done by replotting the function over a smaller domain. This is recommended if you want the most accuracy. However, the `zoom` function allows us to quickly (if not roughly) identify these points of interest.

The zoom feature may be turned on by entering the command >> `zoom on` and turned off by entering the command >> `zoom off`. Alternately, on the figure window are two icons for setting the zoom features. These are shown in Figure 1.

Figure 1: Icons in graph window to adjust `zoom`

Clicking the + magnifying glass allows you to zoom into a figure. When you click the mouse anywhere in the figure, the figure will be replotted using 1/2 of each axis; in effect, you zoom in by a power of 4. If you want more than this, you can use the mouse to drag out a box on the figure. When the mouse button is released the highlighted window area is shown. The − magnifying glass reverses the zoom style.

Exercise 1:
To illustrate, type in the following commands, which plot the function $f(x) = \sin(x^2)$ over the interval $[0, \pi]$.

```
>> x = linspace(0, pi);
>> y = sin(x.^2);
>> plot(x,y); grid
>> zoom on                       % just zoom works here too
```

Now clicking in the figure window will cause the graph to be redrawn over a smaller domain.

a. Click near the smallest **positive** x in this interval where f(x)=0. (The positive x-value where the graph first crosses the x axis). What is the length of the interval shown on the y axis?
    (1) Answer: _____

b. Click twice more near this point of first intersection. Estimate (no more than two decimal points) from the zoomed in graph the numeric value of the x for which $f(x) = 0$.
    (2) Answer: _____

c. Click the − magnifying glass and zoom out. Then the + magnifying glass to zoom in. This time on the second intersection point. Click near it 3 times. Estimate this value of x for which $f(x) = 0$.
    (3) Answer: _____

3 Identifying an appropriate domain

MATLAB can't think for us, but it can help us "think easier." Case in point, we're often interested in plotting a function on only part of its domain. This may be where a function makes sense for a physical problem such as non-negative time, or non-negative area. Or, it may be where we can identify one cycle of a periodic function. We can analyze the function to do this, or we can use MATLAB to explore some graphs and from these decide upon an appropriate domain.

Example 1:
A baseball is thrown from center field towards home plate. Its height, y, varies as a function of time, t. It is known from high-school physics that a model for this is given by the projectile motion formula

$$y = y_0 + v_0 t - \frac{1}{2}at^2,$$

where y_0 and v_0 are the initial height and upward velocity, and a is the constant of acceleration. In metric units of meters and seconds, we will assume $a = 10$ instead of the more accurate 9.8 m/s^2.

Suppose a ball is thrown with an initial velocity of $v_0 = 10$ meters per second upward from an initial height of $y_0 = 2$ meters. On what realistic domain will $y(t) \geq 0$?

Mathematically, $y(t)$ is a parabola which opens downward as the t^2 coefficient is negative. We could use formulas to find a parabola's zeroes, but using MATLAB we can have fun exploring the graph just as easily. First, we make an initial plot of the data over the interval $[0, 10]$. Why that? Well time must be non-negative so $t \geq 0$ is natural. As for 10, since this is supposed to model a ball being thrown, 10 seconds should be long enough, if the model is realistic.

```
>> t = linspace(0,10);
>> y = 2 + 10*t - (1/2)*10*t.^2;
>> plot(t,y)
```

If you make this graph, you'll see that 10 seconds was too long. (Why). Let's replot using an interval of $[0, 2]$.

```
>> t = linspace(0,2);
>> plot(t,2 + 10*t - (1/2)*10*t.^2)
```

(We combined the definition of y directly into the plot function.)

Now our interval is too small.

Exercise 2:
Replot the ball problem until you can find a good estimate for the value of b so that $y(t) \geq 0$ on the interval $[0, b]$, and is negative for $t > b$. What is the value of b that you found?
(4) Answer: _____

Example 2:
Now suppose we want to plot one period of this periodic sine function:

$$f(x) = 50 + 30\sin(\frac{2\pi x}{365})$$

We could try to remember the rules for the shape of sine curves based on the general format $y = d + \sin(bx + c)$. This is best, but do you remember? What we can do **without having to remember** is to plot this formula over a few different viewing windows until we can figure it out. Here is an example with comments
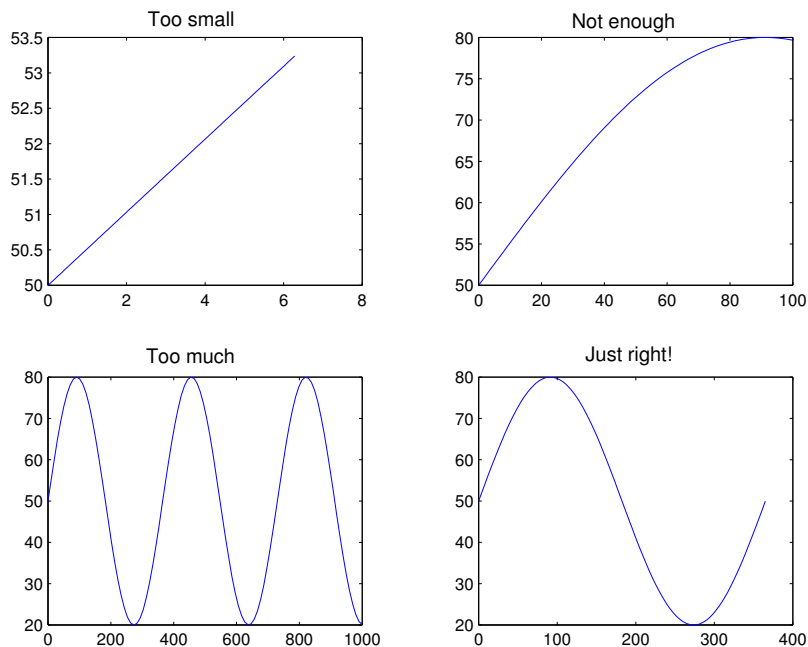

Figure 2: Different viewing windows to frame one period of a sine function.

```
>> x=linspace(0,2*pi); plot(x,50+30*sin(2*pi*x/365))
%% [0,2pi] is always a good start, but it is too small. Try a larger domain
>> x=linspace(0,100); plot(x,50+30*sin(2*pi*x/365))
%% Still too small. Be bold let's see try 1000
>> x=linspace(0,1000); plot(x,50+30*sin(2*pi*x/365))
%% Oops, too big now
>> x=linspace(0,365); plot(x,50+30*sin(2*pi*x/365))
%% just right. Oh yeah T=(2*pi/b)!
```

That doesn't take too long, as we only need to change one number and rerun the line. To take advantage of the "up arrow" trick, we put the commands on one line and combined the function evaluation with the `plot` function.

Exercise 3:
Find the suggested values by plotting the graphs until you can figure out the answer.

a. Repeating the above example, find one period of the function

$$f(x) = 120\sin(120\pi x).$$

(This is a bit tricky! Keep plotting until you clearly get one period shown in the viewing window.)
(5) Answer: _____

b. When does $f(x) = 25 - x - \sin(x)$ cross the x-axis? (Guess an answer to within 1 decimal point.)
(6) Answer: _____

4 Plotting functions with vertical asymptotes

Plotting functions with vertical asymptotes can cause troubles. For instance, suppose we want to plot the function $f(x) = 1/\sin(x)$ over the interval $[0, \pi]$. A first attempt may look like this:

```
>> x = linspace(0,pi);
>> plot(x, 1./sin(x))
```

If you make this graph, you'll find that it doesn't look like what you would expect. In this case the function has a vertical asymptote at both endpoints 0 and π. This means that when the x values are close to the endpoints, the corresponding y values get to be very large. This is why the y-axis is labeled using scientific notation indicating 15 zeroes after the numbers. This makes this graph worthless for answering questions about $f(x)$. The remedy is to replot, only this time avoiding the asymptotes:

```
>> delta = 0.1
>> x = linspace(delta, pi-delta)
>> plot(x, 1./sin(x))
```

This is much better, allowing us to see the shape of the graph. We used a variable `delta` so that if we wanted to, it is easy to make changes.

Exercise 4:
From its graph, estimate the minimum value of the function

$$f(x) = \frac{5}{\cos(x)} + \frac{8}{\sin(x)}$$

over the interval $(0, \pi/2)$.
    Just plotting with x values given by

```
>> x = linspace(0,pi/2)
```

will cause the same problems as above. Make a plot, judiciously avoiding the vertical asymptotes, and then find the minimum value (y value) of the function on this interval.

(7) Answer: _____

Exercise 5:
Although it isn't an asymptote, this next problem has the same problem with the default y-axis scale being too big to accurately see other features of the graph.

Let

$$f(x) = x^5 - 225x^3 - x^2 + 225.$$

Our goal is to find the three real roots (zeroes) of this polynomial.

a. First plot the graph on the interval $(-17, 17)$. What are the values of the x intercepts that you can clearly see.
   (8) Answer: _____

b. Is 0 the other root? From the graph over $-17, 17$ it seems plausible. Explain why you know that 0 is not the other real root for this polynomial.
   (9) Answer:

c. Replot the function over an appropriate domain to find the third real root:
   (10) Answer: _____

5   Plotting Several Functions on the Same Graph

Often we wish to compare two functions. We may want to see which is bigger? Which grows faster? Or, we may want to see when they are equal. In order to do this we need to learn how to put multiple graphs into one figure. We illustrate two ways that this can be done:

a. You can draw all the graphs at once using `plot`, as in `plot(x1,y1,x2,y2)`.

b. You can use the `hold` function to cause MATLAB not to draw a new plot each time, but rather to hold the current plot so that new lines may be added to it.

Example 3:
Plot the functions $f(x) = 4\cos x$ and $g(x) = \cos 4x$ together over the interval $0 \le x \le 2\pi$.
Solution: We first create pairs of arrays as if we were to graph each function separately.

```
>> x = linspace(0,2*pi);
>> y1=4*cos(x);              % this is the first function
>> y2=cos(4*x);              % the second function needs a different name
>> plot(x,y1,x,y2), grid     % plot(x,y) is one graph. This makes two
```

MATLAB creates a frame in which all function values fit, and then plots the graph of each function. Different colors are used for each function.

If we prefer, `plot(x,y1,x,y2)` can be replaced with:

```
>> plot(x,y1)
>> hold on          % next plot will be on top of the current one
>> plot(x,y2)
>> hold off         % turn off the hold. Next graph opens a new window
```

The `hold on` command instructs MATLAB not to create a new plot window with subsequent `plot` calls. By entering `hold off` at the end of the example, we allow new plot windows to be created.

Exercise 6:
Consider the functions of the previous example, $y_1 = 4\cos(x)$ and $y_2 = \cos(4x)$, make the graphs and use them to answer the following:

  a. Which function is oscillating more rapidly?
     (11) Circle one:
     1. $\cos(4x)$
     2. $4\cos(x)$

  b. Which function has the larger amplitude?
     (12) Circle one:
     1. $\cos(4x)$
     2. $4\cos(x)$

Exercise 7:
Let $f(x) = \sqrt{x}$. Plot $f(x)$ and $g(x) = f(x) - 5$ together over the interval $[0, 10]$. The graph of $g(x)$ is the graph of $f(x)$ shifted ...
(13) Circle one:
1. **left** by 5
2. **down** by 5
3. **right** by 5
4. **up** by 5

Exercise 8:
We wish to graph a function and its tangent line. Graph both the function $y = \sqrt{x}$ over the interval $[0, 9]$ and the line through the point $(4, 2)$ with slope 1/4.
    (14) **Attach your graph to the worksheet.**

5.1   Plotting Functions Defined by Several Equations

Example 4:

    Cell phone plans often have a flat rate for a fixed amount of minutes and then a per minute charge for each minute over the fixed amount. Mathematically, this is described by a function with two different definitions depending on which part of the domain – if the value is less than the fixed amount or more.
    For instance, if a company offers 500 minutes of calling for \$25 and thereafter it is 25 cents per minute, a function that describes the amount you owe based on the number of minutes is

$$f(x) = \begin{cases} 25 & \text{if} \quad 0 < x \leq 500 \\ 25 + .25(x - 500) & \text{if} \quad 500 < x \end{cases}$$

(Neglecting the onerous extra charges and taxes of course.)
    To plot this function (Figure 3), we handle each part of the function definition separately.

```
>> x1 = linspace(0,500);
>> y1 = 25 + 0*x1;              % note the funny 0*x1
>> x2 = linspace(500,1000);
>> y2 = 25+.25*(x2-500);        % if x>500 we use this line
>> plot(x1,y1,x2,y2),grid       % plot both
```

    This example shows a funny trick to plot the horizontal line y2 = 25+0*x1 instead of simply y2 = 25. Try plotting both ways to see why we did so.

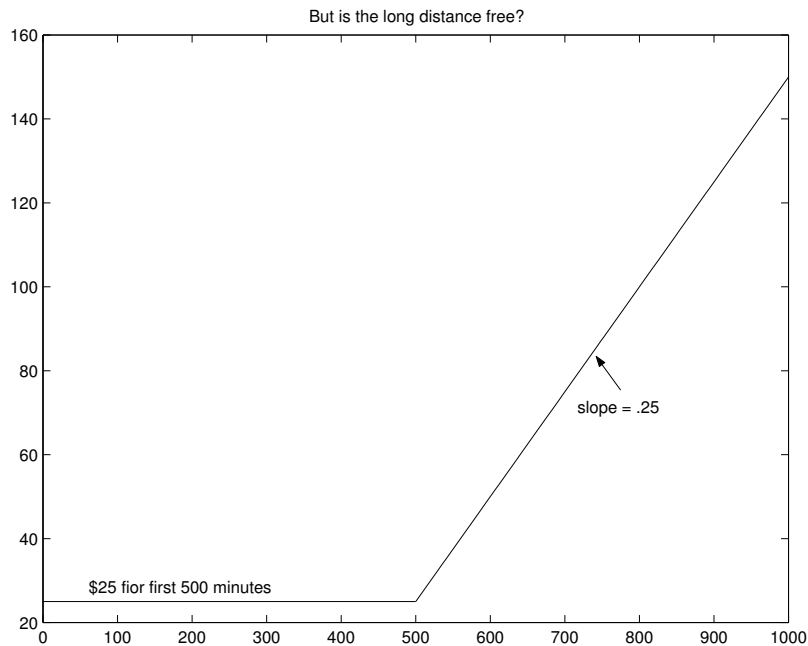Figure 3: Plot of phone charges

Exercise 9:
Plot a graph that describes the following calling plan: You pay $35 for the first 1,000 minutes. After the first 1,000 minutes you pay $0.30 per minute. Draw a graph for usage between 0 and 2,000 minutes.

    a.   What commands did you use?
        (15)

    b.   (16) **Attach your graph to the worksheet.**

6   Adding additional information to a graphic

We may want to add additional information to a graphic. This may be done when we issue a `plot` command, or through the dialogs offered by the figure window.

    Here are few things that you can do:

Setting a plot to have a certain color When you plot two vectors you can set the color of the line that is drawn by adding it as a third argument, such as

```
>> plot(x,y,'g')                        % 'g' for green.
```

Some other basic colors are 'b' for blue, 'r' for red, 'y' for yellow.

Plotting points, not line segments You can plot the two vectors as just points by specifying the type of point you want drawn, as the color is specified above. For instance

```
>> plot(x,y,'.')                 % plots with dots, doesn't connect line
```

Other characters are 'o' for circles, '+' for plus signs, 'x' for crosses, 'd' for diamonds, and 's' for squares.

Adding text to a figure Check the *Figure Pallette* dialog under the View menu. You may have to maximize the figure window. You will see, as in Figure 4, a variety of items for annotating your graph. You can label your functions by adding text with an arrow. Try the *text arrow* dialog. Just point, 'drag-and-drop', then type your text. Alternately, use the gtext command from the Command Prompt. (Type help gtext for usage.)

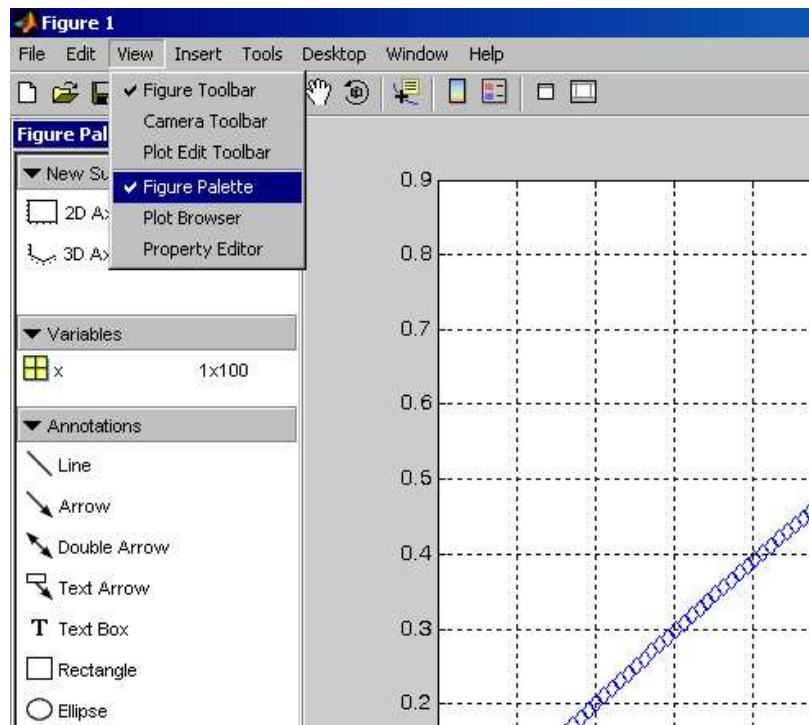Figure 4: `Figure Pallette` menu for adding text, arrows and lines to a figure.

Labeling the axes The Axes Properties dialog under the Edit menu in the figure window allows you to add a title, and edit your axis labels, among other things.

   This next set of exercises uses the topics above with an example of fitting competing mathematical models to some data to try and predict a future value.

Exercise 10:
Suppose your salary was $42,200 in 2003 and $47,500 in 2005. You are interested in your salary in the year 2011, but of course you would like to know this **now**. What do you do? Well, you can try a mathematical model. There are two common ones:

Linear model This model says your salary will go up by a fixed amount each year (the slope). The formula is of the form

$$s = s_{2003} + m(t - 2003),$$

   where t is the year and s_{2003} is the amount you make in the year 2003 which is $42,200.

Exponential model This model assumes your salary increases by a fixed **percentage** each year (not amount as in the linear model). The general formula is of the form

$$s = s_{2003}e^{r(t-2003)}$$

   where r is figured out from the data. We'll help you out. For this problem r is

$$r = \log(\frac{47500}{42200})/2 = 0.0592...$$

   Assume for now that your salary follows a "linear growth" pattern.

   a. What MATLAB command calculates the salary values, s as a function of t?
       (17) Answer:

   b. Assuming linear growth continues through the year 2011, make a graph to display your salary from year 2003 to 2011. After the plot command, label the axes so that the x axis says "year", the y says "yearly salary," and the title says "My salary."

   c. Use the graph to predict your salary in 2011.
       (18) Answer: _____

Now, suppose you are using the exponential model.
On the **same** graph as your linear model (using `hold`), make a graph of the exponential model between the years $t = 2003$ to $t = 2011$.

Exercise 11:

a. On your graph, label the exponential model with "exponential" and an arrow, and the linear model with "linear" using the icons in the figure window menubar.

b. Use the exponential model to estimate your salary in the year 2011.
   (19) Answer: _____

c. The difference between the linear model and exponential model is how they predict the the amount your raise will be each year. One predicts that your salary will increase by a fixed amount each year. The other that your salary will increase by a fixed **percent** each year.

   Which model predicts that your raise will be a fixed percent each year?
   (21) Circle one:
   1. linear model **2.** exponential model

d. What is the fixed percent each year that is predicted?
   (22) Answer: _____

Exercise 12:
On the same graph, mark the two points $(2003, 42200)$ and $(2005, 47500)$ using diamonds (the `'d'` argument to `plot`).

Exercise 13:
On the same graph. Draw the line $s = 60,000$. Your graph should have 3 functions on it now: the linear model, the exponential model and the flat line $s = 60,000$.

a. In what year does the linear model predict that your salary will be at least $60,000?
   (23) Answer: _____

b. In what year does the exponential model predict that your salary will be at least $60,000?
   (24) Answer: _____

c. Attach your graph (all 3 functions and annotations).
   (25) **Attach your graph to the worksheet.**

Exercise 14:

**(Optional:for extra credit)**

The radius r of the smallest circular paper plate that holds a pizza of a given radius R=5 inches and central angle θ, $0 < \theta \le \pi$, is given by the function $r = f(\theta)$ where

$$r = f(\theta) = \frac{R}{2\cos\frac{\theta}{2}} \quad \text{if } 0 < \theta \le \frac{\pi}{2}$$
$$r = f(\theta) = R\sin\frac{\theta}{2} \quad \text{if } \frac{\pi}{2} < \theta \le \pi.$$

Following sketches to be done by hand:

a. Sketch the pizza in its plate for $\theta = \pi/6$ (a 30 degree wedge). Find r for $\theta = \pi/6$.

b. Sketch the pizza in its plate for $\theta = \pi$ (half the pizza). Find r for $\theta = \pi$.

c. Determine the function $f(\theta)$ for the domain $\pi \le \theta \le 2\pi$.

d. Explain how to obtain the formula above for the function $f(\theta)$ on $(0, \pi)$. Hint: Sketch lots of examples.

e. Use MATLAB to create and print a graph of $f(\theta)$. (printed graph to be submitted separately to professor.)
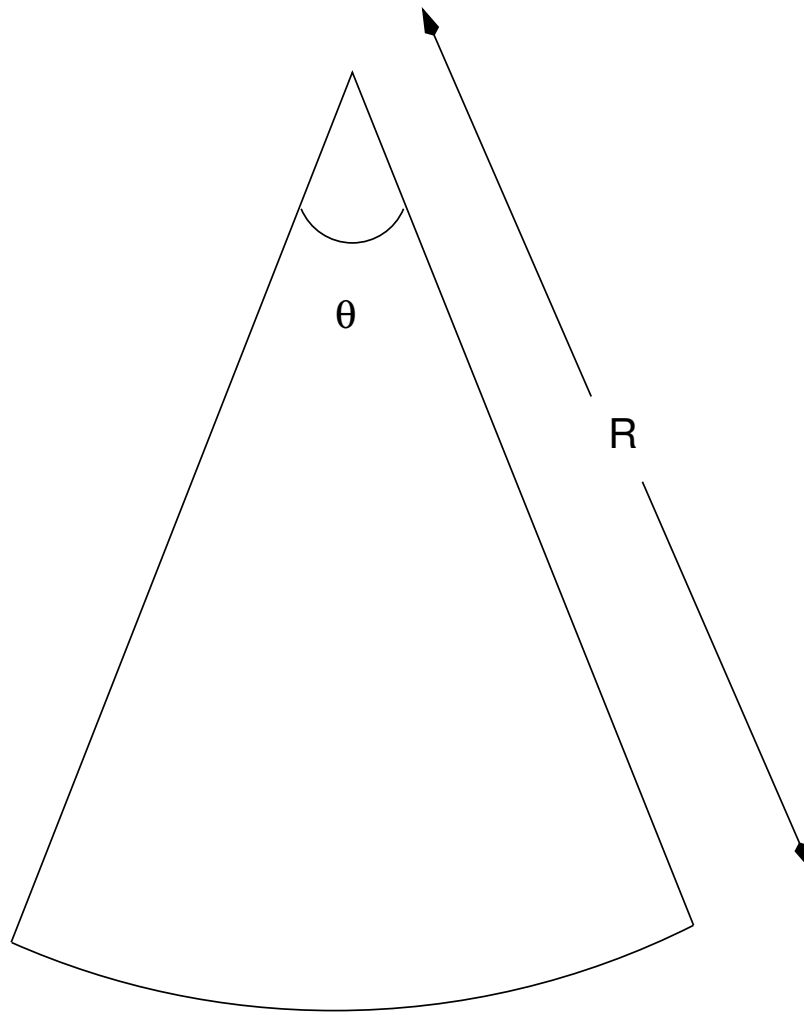
A wedge of pizza. (Yum!)

MTH229

The College of Staten Island
Department of Mathematics

Graphical Solutions to Equations

1 Introduction

In mathematics, many problems are solved by finding when a function is equal to 0, or when two functions are equal. Notationally, this means we solve equations like

$$f(x) = 0, \qquad \text{or} \quad f(x) = g(x).$$

Sometimes these problems can be difficult to do algebraically, but we can find good approximations to the answers graphically. The technique we will use is to plot the function(s) in the appropriate viewing window and then use the `zoom` function to find our answer. When we are solving $f(x) = 0$, we zoom in on where the function crosses the x axis. Using the `grid` function is useful here. The solutions here are called zeroes of the function $f(x)$. When $f(x)$ is a polynomial, we call the zeroes roots.

When we are solving $f(x) = g(x)$, we zoom in on the points where the two functions cross. The x value will be the solution we seek. These problems may also be solved by plotting $f(x) - g(x)$ and looking at the zeroes. You should get the same answers.

1.1 New MATLAB topics

a. Graphically finding intersection points

b. Using built in functions to find zeroes

1.2 New MATLAB commands

a. `roots(p)` – find the roots of a polynomial `p` $= [a_n \quad a_{n-1} \quad \cdots \quad a_2 \quad a_1 \quad a_0]$

2 Finding solutions to equations graphically

We illustrate the graphical solution of equations in the following example.

Example 1:
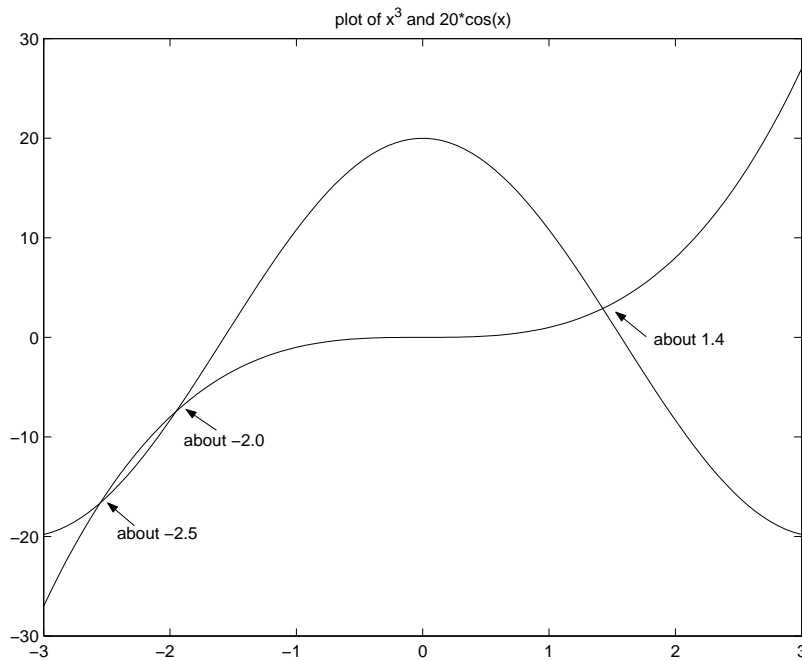Find all solutions of the equation

$$x^3 = 20 \cos x$$

Figure 1: Plot of x^3 and $20\cos(x)$. The solutions to the equation $x^3 = 20\cos(x)$ are the x values of the intersection points.

Step 1. Plan you actions We will graph both $f(x) = x^3$ and $g(x) = 20\cos x$ and look for intersection points. Alternatively, you could plot $f(x) = x^3 - 20\cos(x)$ and look for its zeroes.

To find an appropriate viewing window, we notice that when $x = 3$ we have $x^3 = 27$ which is already more than 20, the largest $20\cos(x)$ can be. By symmetry, we will choose the interval $[-3, 3]$ for our initial plot (Figure 1). If this isn't large enough, or is too small we will make an adjustment and replot.

Step 2. Make the plot(s) The graph is created with these commands

```
>> x = linspace(-3,3);
>> y1 = x.^3;
>> y2 = 20*cos(x);
>> plot(x,y1, x,y2)
```

You could also use `hold` here and make the plots incrementally.

Step 3. Investigate the graph Looking at the graph, we can see that the solutions are about -2.5, -2.0, and 1.4, but we can do better by zooming in around these intersection points.

Step 4. Zoom in Let's find the value of the root near 1.4. First we check if 1.4 is very close by looking at the difference between the two functions:

```
>> x = 1.4
x = 1.4000
>> x^3 - 20*cos(x)
ans = -0.65534
```

We are not very close. We turn the `zoom` feature on using the magnifying glass icon, or the zoom function and then click near the intersection point. The plot is redrawn with half the x and y axes. We see the answer is closer to 1.42. Clicking again, we see that a value of 1.425 seems about right. We check to see how close we are

```
>> x = 1.425
x = 1.4250
>> x^3 - 20*cos(x)
ans = -0.0120
```

It is not exact (1.4255 is closer), but we have to be careful when we zoom in too close, as our graph may not accurately portray the actual function. Recall we have plotted 100 points evenly spaced between -3 and 3 so that roughly there are 16 points between 1 and 2. After zooming in this closely, the plot is made up of just a few line segments, whereas the functions' graph is actually a curve. To get more accuracy, it is best to replot with more points in the neighborhood of the answer you are looking at.

Although zooming is valuable to quickly find pretty good answers, there are other methods from numerical analysis which are more accurate. In a later project we explore one called Newton's method.

Exercise 1:
Use a graph of $(x - 2)^2 = 4\sin(x)$ to find solutions to the equation valid to 2 decimal points:
(1) Answer: _____

Exercise 2:
Use the zooming technique to find solutions of

$$50 + \sin x = 2x.$$

which are valid to at least two decimal places.
Hint: Try to estimate the value of $50 + \sin x$. This will give you an idea in which x interval are the possible solutions!
(2) Answer: _____

Exercise 3:
Folklore is that exponential functions grow faster than polynomial functions. Although true, you need to be careful about how you interpret this statement, as this exercise shows.

Consider the functions $z_1 = e^x$ and $z_2 = x^4$. Plot them together on the interval [0,4].

a. From their graphs, how can you determine which graph is the exponential and which is the polynomial?
   (3) Circle one:
   1. polynomial functions grow faster than exponential functions
   2. Exponential functions grow faster than polynomial functions
   3. For different values of x, I can evaluate z_1, z_2 and determine which is larger.

b. Find the value of x (to two decimal places) for the point of intersection by zooming on the zero of $f(x) = e^x - x^4$. (or by zooming on the intersection point of the functions $z_1 = e^x$, $z_2 = x^4$.)
   (4) Answer: _____

   On this graph, x^4 is larger than e^x from the intersection point to $x = 4$. Experiment to determine how large a value of x is needed for the exponential to catch up to x^4. Then find the second intersection point. (correct to three decimal places.) This one is larger than 4. In fact, you now have found two intersection points (x_1, y_1), (x_2, y_2). (where $x_1 < x_2$) Up to x_1 the function e^x is bigger, from x_1 to x_2 the function x^4 is the bigger. What happens after x_2?

c. What is the x-coordinate of the second intersection point?
   (5) Answer: _____

d. What happens to the behavior of z_1 and z_2 after the second intersection point?
   (6) Circle one:
   1. e^x grows faster
   2. x^4 grows faster
   3. they grow at the same rate
   4. e^x grows faster, but for increasingly large values of x, x^4 catches up to e^x again.

   Vertical asymptotes can be a real impediment to finding roots, as this example illustrates.

Example 2:
Find any zeroes and the minimum value of the rational function

$$f(x) = \frac{x+2}{x^2}.$$

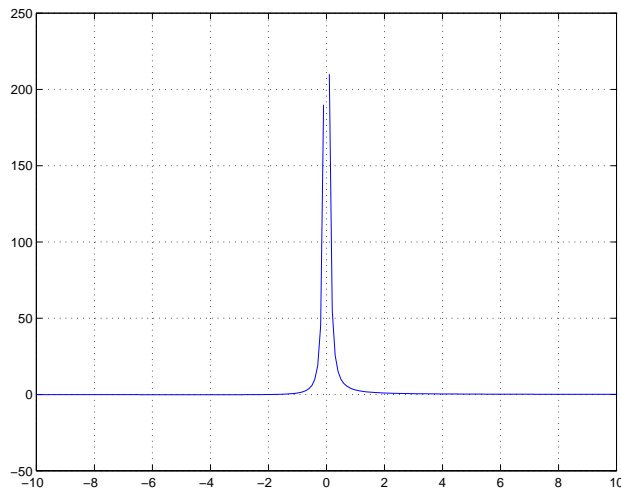   We first graph the function over initial interval $[-10, 10]$ producing Figure 2.

Figure 2: The function $f(x) = (x+2)/x^2$ illustrating a vertical asymptote.

The figure illustrates that the vertical asymptote at $x = 0$ gets in the way of finding the two answers we want—the zeroes and the minimum value. Let's try again using a little thought before we chug ahead with the computer.

Recall that rational functions are 0 only when the numerator is, that is when $x + 2 = 0$, or $x = -2$. The function goes from negative to positive there, so the minimum must be located to the left of -2. As this function has a horizontal asymptote of $y = 0$ there must be a minimum to the left of -2, and no minimum to the right of 0 where the function is positive. You'll do the rest in the following exercise.

Exercise 4:

    a. Find the x-coordinate for where $f(x) = (x+2)/x^2$ achieves its minimum value.
       (7) Answer: _____

    b. What interval on the x-axis did you use to make you plot window?

       (8) Answer:

3   Finding roots of polynomials

Polynomial functions are extensions of linear and quadratic functions ($a \neq 0$):

$$f(x) = ax + b, \quad \text{(linear)} \quad f(x) = ax^2 + bx + c \quad \text{(quadratic)}.$$

For these two types of functions we can solve $f(x) = 0$ easily. For the linear case the single solution is $-b/a$. For the quadratic case the quadratic formula produces two answers. These answers are the roots of the polynomial.

   The quadratic formula produces two roots, although many possible cases exist: two distinct real roots, double roots, or two complex-valued roots that come in conjugate pairs. There are always no more than 2 real roots.

   A general polynomial of degree n may be written as

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots a_2 x^2 + a_1 x^1 + a_0.$$

For an nth degree polynomial there are n, possibly complex, roots counting multiplicities. But they need not all be real roots. However, there is not always a formula for solving for these roots.

   However, MATLAB has a built-in function, called `roots`, that numerically solves for the roots, which is faster and more accurate than the zooming technique. To use `roots`, you must code the polynomial $f(x)$ like a vector. We take each coefficient (there are $n + 1$) and enter them in order as follows:

   The polynomial $f(x)$ is represented with
>> p = [a_n a_{n-1} $\cdots$ a_2 a_1 a_0]

   Then the polynomial equation $f(x) = 0$ is solved with the command

```
>> roots(p)
```

   The only subtlety is that we don't usually write the terms whose coefficients are 0, but we must do so when entering in the polynomial into MATLAB. So, for instance

$$\begin{aligned}
&\texttt{[3 2 1]} &&\text{represents } f(x) = 3x^2 + 2x + 1, \\
&\texttt{[3 2 1 0]} &&\text{represents } f(x) = 3x^3 + 2x^2 + x, \\
&\texttt{[3 2 0 1]} &&\text{represents } f(x) = 3x^3 + 2x^2 + 1, \text{ and} \\
&\texttt{[3 0 0 2 1]} &&\text{represents } f(x) = 3x^4 + 2x + 1.
\end{aligned}$$

Example 3:
Use the `roots` function to find the roots of $f(x) = 2x^3 + 6x^2 - 4x - 5$.

```
>> p = [2 6 -4 -5]
p =
   2   6  -4  -5
>> roots(p)
```

```
ans =
  -3.3732
   1.0675
  -0.6943
```

We get 3 real roots. We could verify this by plotting the polynomial over the interval $[-4, 2]$ and observing the graph crosses the x axis three times. Instead, we illustrate that these are roots, by evaluating the polynomial at these values:

```
>> x = roots(p)                 % store the values
>> 2*x.^3 + 6*x.^2 -4*x -5      % evaluate the polynomial for roots
ans =
    1.0e-13 *
     -0.3197
      0.0444
           0
```

These values are very close to 0, as they are multiplied by `1.0e-13`. This is MATLAB's scientific notation for 10^{-13}.

Exercise 5:

a. Let $f(x) = x^3 - 7x^2 + 2x + 9$. Solve the cubic equation $f(x) = 0$. Find all of its roots correctly up to 4 significant digits.
    (9) Circle one:
    1. 6.6 , 1.1 -0.7
    2. 6.4766, 1.4692, -0.9458
    3. 6.7053 , 1.3259 , -0.8259
    4. 0.0010, 1.0100, 7.5902
    5. 6.5806 , 1.1062, -0.6868

b. Now find all solutions to $x^3 + 2x + 4 = 0$ (Note that the coefficient of x^2 is now 0).
    (10) Circle one:
    1. 0.6641, -0.6640, -1.3283
    2. 1.8230, -1.8230, -1.3283
    3. $0.5898 \pm 1.7445i \quad -1.1795$
    4. $1.8230 \pm 0.6641i$, -1.3283

This last exercise illustrates that both means of finding zeroes of functions can be useful.

Example 4:
The flight of an arrow through space can be modeled with or without air resistance. How do the models differ?
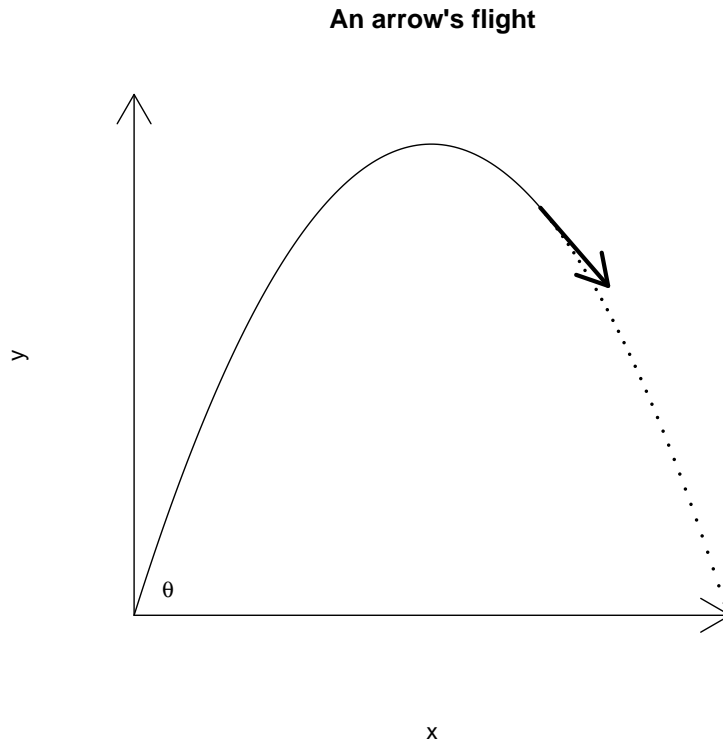
An arrow's flight

Figure 3: Flight of an arrow without air resistance.

When an arrow encounters no air resistance the laws of projectile motion from high-school physics apply. Using x for the horizontal distance traveled, $f(x)$ for the height of the arrow when it is x units away, and θ for the initial angle, we have this model for the trajectory of an arrow without air resistance:

$$f(x) = \tan\theta x - 16\left(\frac{x}{200\cos\theta}\right)^2$$

When there is a drag on the arrow proportional to its velocity (k is the proportion factor), the height of the arrow is given by:

$$g(x) = \tan(\theta)x + \frac{32}{k200\cos\theta}x - \frac{32}{k}\log\left(\frac{200\cos\theta}{200\cos\theta - x}\right).$$

For both models, the **range** of flight is the time that $y \geq 0$. (When $y \leq 0$, the definition of both $g(x)$ and $f(x)$ should be set to 0.) The range can be written as $[0, b]$.

First, we investigate an arrow's flight without wind resistance.

Exercise 6:

a. Let $\theta = \pi/4$. Look carefully at $f(x)$, it is a quadratic polynomial in x. Rewrite $f(x)$ so that the coefficients appear as (careful with the scientific notation)

$$f(x) = ax^2 + bx + c.$$

Now represent this polynomial in MATLAB, as in [a b c]. What are the values:

(11) Answer:

b. Use your previous answer and the roots function to find the range ($[0, b]$) of an arrow when shot at an angle of $\pi/4$. Specify the range in terms of its endpoint b.

(12) Answer: _____

Next, we consider the model with wind resistance. To keep matters simple, we assume $k = 1/2$. Then our formula for the trajectory when $\theta = \pi/4$ becomes, in MATLAB,

```
>> theta = pi/4;
>> k = 1/2
>> a = 200*cos(theta);
>> b = 32/k;
>> g = (tan(theta)+ b/a)*x - b*log(a ./(a-x));
```

provided you have defined values for x.

Exercise 7:

a. Plot various graphs of the $g(x)$ until you find the range of g, $[0, b]$. Enter the value of b with at least 1 digit to the right of the decimal point. (Remember, arrows don't bounce up – this mathematical model is only valid until the arrow first hits the ground.)
    (13) Answer: _____

b. Now make a plot containing the trajectories of both models. Label the individual plots.
    (14) **Attach your graph to the worksheet.**

c. From your graph estimate the maximum height of the arrow if there is no wind resistance.
    (15) Answer: _____

d. From your graph estimate the maximum height of the arrow if there **is** wind resistance.
    (16) Answer: _____

4   Extra credit problem

(To be handed in on a separate sheet of paper.)

It can be shown that the maximum range for the arrow if there is no wind resistance happens when $\theta = \pi/4$. However, this is not the case for the model with wind resistance. By exploring around with different values of θ, find a value of θ that maximizes the trajectory.

MTH229

<div align="center">

The College of Staten Island
Department of Mathematics

Investigating Limits in MATLAB

</div>

1 Introduction

In this lesson, we learn how to investigate limits of functions using MATLAB. First a review of some key facts about limits.

For **continuous** functions it is easy to find a limit because

$$\lim_{x \to c} f(x) = f(c).$$

This is sometimes called "plugging in," as the limit at c is simply the value of the function at c. Graphically, we find the limit at c by observing the corresponding y value at c

Other functions are not continuous at a point c, but would be if the function could be redefined at c. These values of c may be called removable singularities. Although the function is not defined at c, we can still find the limit graphically. Simply follow the graph over to where the function should be defined at c to make it continuous.

Some functions have only right or left limits at a point c. This means we don't look on both sides of c to investigate the functions behavior, rather only one side. Of course, a function has a limit only if it has a left and right limit and the two are the same. Left and right limits may be investigated graphically, by plotting the graph of the function on just one side of c.

We can also investigate limits using a table. The limit of $f(x)$ at c is intuitively the value that $f(x)$ approaches as x approaches c. Often we can take x close to c and then $f(x)$ will be close to the limit $\lim f(x)$ as $x \to c$. We can make a table of values getting close to c and a corresponding table of function values, and look for trends.

This project will investigate limits both graphically and numerically. As well, as a useful aside, we will learn how to define functions in MATLAB so that our work can be simplified.

1.1 New MATLAB topics

a. Function files (script files too)

b. `format [long short]` – control the format of MATLAB's answers

2   Finding limits with MATLAB

With MATLAB we can't use the ϵ–δ definition of a limit, but rather rely on what the limit of a function tells us, namely as x approaches c the function approaches the limiting value.

As an example, we illustrate how to find the following fundamental limit using MATLAB:

$$\lim_{x \to 0} \frac{\sin x}{x}$$

First we note that this function, $f(x) = \sin(x)/x$ is continuous everywhere except at $c = 0$, so trying to "plug in" will fail. To see how, we try use MATLAB:

```
>> x=0;  sin(x)./x
Warning: Divide by zero.
ans = NaN
```

Hmmm, Not a Number. As expected.

A plot of the function near 0 may be illuminating. Although we only need to plot the function near 0, we show in Figure 1 the interval from -2π to 2π so that we can see that this function oscillates like a damped sine curve, except near 0 where the limit can be seen to be 1.

```
>> x = linspace(-2*pi,2*pi)      % numbers around 0.
>> plot(x,sin(x)./x)
```

How come there is no error message when $x = 0$? This is because 0 is not one of the values in x. We need to take an odd number of points to make that happen, as you can verify if you were to use `linspace(-2*pi,2*pi,101)` for x. What happens instead of an error is that the points nearby 0 get connected with a line. As this point of discontinuity is a removable singularity, the graph appears to be continuous, as though $f(x)$ were defined to be 1 at 0.

2.1   Finding limits using tables

How might we have found this value numerically? The basic idea is that to investigate

$$\lim_{x \to c} f(x)$$

we look at values of x "close" to c and compare what the corresponding $f(x)$ values get "close" to.

To take x values close to c can be done by hand as with

```
>> x = [.1 .01 .001 .0001 .00001];
```

The corresponding values of $f(x)$ are then

```
>> y = sin(x) ./x;
```

Graph of sin(x)/x. The limit at 0 is 1.

Function is undefined
at x=0. But is continuous
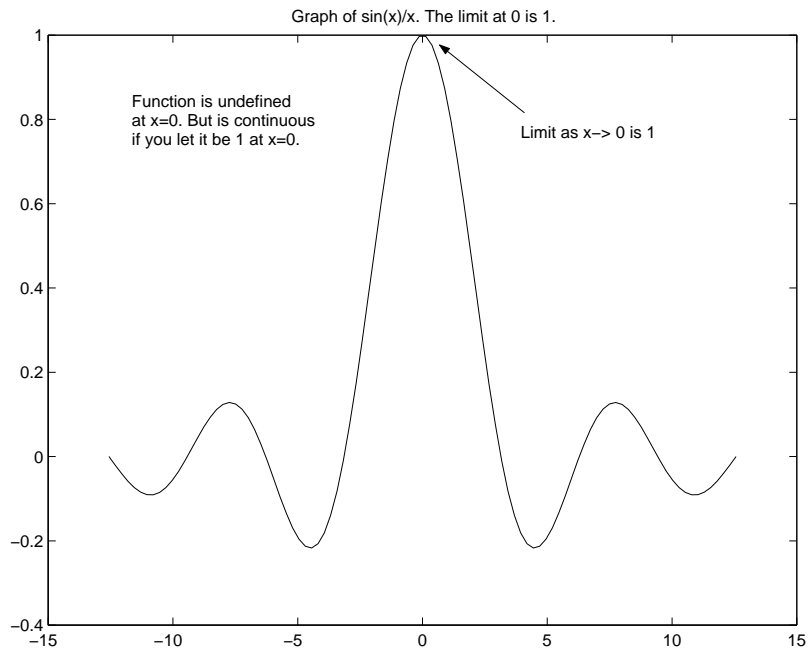if you let it be 1 at x=0.

Limit as x--> 0 is 1

Figure 1: Plot of $f(x) = \sin x/x$ showing a limit of 1 at $c = 0$.

To see them in a table format for easy comparisons we use the following notation:

```
>> format long  %14 digits to right of decimal point
>> [x; y]'          % a semicolon and a '
ans =
   0.10000000000000    0.99833416646828
   0.01000000000000    0.99998333341667
   0.00100000000000    0.99999983333334
   0.00010000000000    0.99999999833333
   0.00001000000000    0.99999999998333    % y values get close to 1
```

The tables shows that $f(x)$ gets close to 1 as x gets close to 0 from the right. We should check the values from the left side of 0 as well, but in this case we merely note that $f(x)$ is an even function so a left limit at 0 will exist and be the same as the right limit.

You can toggle between showing 4 or more decimal places using the commands

```
>> format short             % show fewer decimal places
>> format long              % show more decimal places
```

One caveat, when looking for convergence, you may not find it if there are numerical instabilities.

Exercise 1:
Use the graphical approach to find the following **right** limit of $f(x) = x^x, \quad x > 0$

$$\lim_{x \to 0^+} x^x$$

What is the value of the limit?
(1) Answer: _____

Exercise 2:

a. Use the numeric approach to find the following limit

$$\lim_{x \to 0} \frac{1 - \cos(x)}{x^2}$$

What is the value of the limit?
(2) Answer: _____

b. Numerically investigating a limit at a point c which is not zero requires us to use points which get close to c. A simple trick allows us to reuse our points which get close to 0. For instance if

```
>> x = [.1 .01 .001 .0001 .00001]; % and
>> c = 1/2;
```

Then we can make points getting close to c from above and below as follows:

```
>> c + x;                        % points above c
>> c - x;                        % points below c
```

Use this to investigate

$$\lim_{x \to \frac{\pi}{2}} \left(\frac{\pi}{2} - x\right) \tan(x).$$

What value for the limit do you get? (Enter no limit as DNE.)
(3) Answer: _____

Exercise 3:

To illustrate what can happen numerically with some functions, we again look at finding the limit

$$\lim_{x \to 0} \frac{1 - \cos(x)}{x^2}.$$

Only we let x get really close to 0. First we use some MATLAB shortcuts to define increasingly smaller values in x

```
>> n = 1:8; x = 10^(-n);        % Small values
```

Now produce a table of the $f(x)$ values for these values of x. Use this to investigate the limit at 0 of the function. How does the numerical instability of the problem show up in the output?

(4)
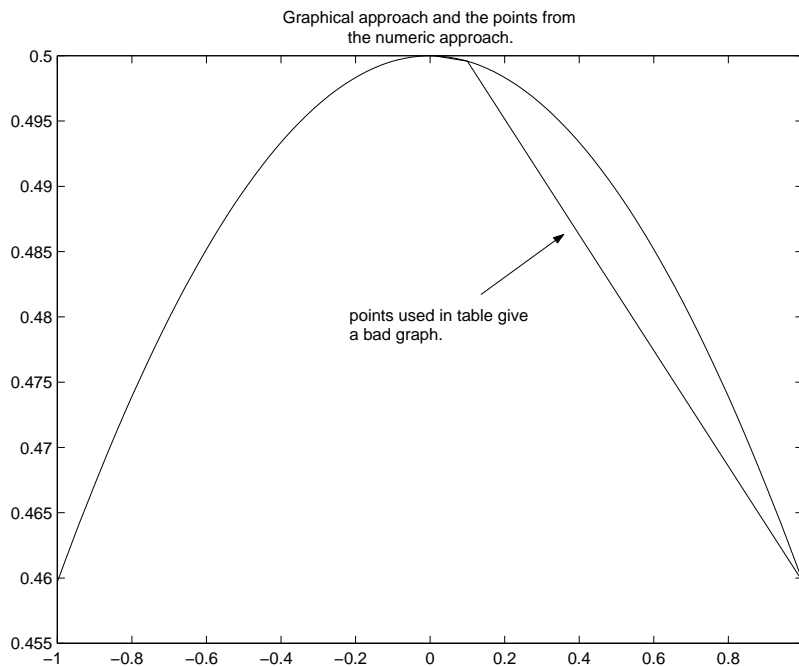
Figure 2: Figure illustrating graphical and numeric approach to solving a limit. Notice the x values chosen for the numeric are bad values for making the graph of the function.

Exercise 4:

Let $f(x) = x \ln(x), \quad x > 0$. Calculate the following right limit with MATLAB as directed:

$$\lim_{x \to 0+} f(x).$$

a. Use a vector of numbers that converge to 0 from the right to investigate numerically the limit as x goes to 0. What is your numeric estimate for the limit?

(5) Answer: _____

b. Enter in your table of values here. (Use the syntax `[x;y]'`.)

(6)

c. Plot a graph of $f(x)$ over $(0, 1)$. Does the graph confirm your limit found from the table of numbers?

(7) Circle one: 1. Yes **2.** No

d. Although the function is only defined for $x > 0$, plot a graph of $f(x)$ over $(-1, 1)$. Does the graph show anything unusual?

(8)

Unfortunately, not all graphs will immediately give us an answer. The following exercise illustrates a possible trouble identifying a limit from the graph and then shows how a theorem from calculus can help us figure out the actual limit.

Exercise 5:

We wish to find the limit of the oscillating function

$$f(x) = x \sin\left(\frac{1}{x}\right)$$

as x approaches 0.

Plot the function $f(x) = x \sin(1/x)$, over the interval $[-\pi, \pi]$ using 100 points for x.

a. Try to `zoom` in on the limit. What happens near $x = 0$ that prevents you from getting the correct answer?

(9)

---

b. Graph the same function using 1000 points on top of your current graph (`hold`) using a different color.

c. The graph with 100 points didn't allow us to zoom in far enough to see the limit as this function oscillates so often that when we zoom in we eventually get just straight lines.

Do you have this same problem using 1000 points?
(10) Circle one: 1. Yes **2.** No

d. Motivated by the squeeze theorem, on the same graph (`hold on`) plot both $y = |x|$ and $y = -|x|$ ($|x|$ is `abs(x)` in MATLAB) What relationship, if any, can be observed between $f(x) = x \sin(1/x)$ and these two lines?

(11) Circle all that apply:
1. $|x| \leq f(x)$
2. $|x| \geq f(x)$
3. $-|x| \leq f(x)$
4. $-|x| \geq f(x)$

e. Graphically estimate the limit as $x \to 0$ What is the limit?
(12) Answer: _____

f. How did graphing the absolute value functions help you find the limit?
(13) Circle one:
1. The mean-value theorem
2. The function is continuous
3. the squeeze (or sandwich) theorem
4. they didn't I just guessed

3   Using Functions

A mathematical definition of a function $f(x)$ is that it is a rule that assigns to each x in its domain a single value y in its range. We might call x the input and y the output. We use functions for many reasons. First they allow us to generically talk about problems. So instead of saying $\lim x^2 - 2x$ we can talk about $\lim f(x)$. This makes it easier to make general statements and to identify the type of problem we may be encountering. As well, functions allow us to simplify our work. Instead of writing out the function definition each time, we can simply refer to it by its name.

Using MATLAB we seek the same abilities. MATLAB provides a mechanism to define functions, we which explain quickly below. More details are provided at the end of the project.

First we break the function into its three pieces: its input, its rule, and its output. Our basic template for defining a function will look like

```
function outputVariable = functionName( inputVariable(s) )
  outputVariable = rule depending on inputVariable(s)
```

Usually the output variable and input variables are simply named, like x or y. For instance, this will define a MATLAB function to evaluate $f(x) = \sin(x)/x$:

```
function y = f(x)
  y = sin(x) ./ x;
```

This function template will define the function $g(x) = 7x^2 \sin(1/(7x^2))$:

```
function y = g(x)
  a = 7*x.^2;
  y = a .* sin(1./a);
```

A few things to note, the first line has g(x) as we want to call this function by the name g. Second, the rule here uses two lines, as the function naturally lends itself to two pieces. This is not an issue, the last definition for y, the output variable, is what gets returned.

The main features of our function template are:

The keyword function The first (non-comment) word is the keyword function which tells MATLAB that we are defining a function.

The name of the function The name that we will use to call this function appears in the first line. In the two examples, the names are f and g.

The definition of the input and output variables The output variable is defined just after the function keyword. We will always use y for the name, but this is only by convention, more complicated return values are possible. The value returned by the function is the last definition of this variable within the body (the rule) of the function.

The input variable(s) are specified after the name of the function and within parentheses. In our examples there is only 1 input variable, more are possible.

The body of the function The function's rule is defined in the body of the function. You can use the input variable names inside the body. You must take care to assign a value to the output variable in the body of your function if you want to return any information. We put semicolons after each line in the body to keep things quiet, otherwise MATLAB will print the value of each line while processing the function.

Now where do we define a function? It would be convenient to do so at the command line, but we can't. Instead MATLAB has a different mechanism involving files on the harddrive. These are called "function" or "script" files, and their extension will always be `.m`. The basic idea is as follows: (cf. Figure 3)

a. We open a file and type in our function template. Suppose the name is `f`. MATLAB has a built in editor for editing functions (like notepad, only better). This editor uses tabs so that many functions may be edited at once. It is found under the `File` menu of the main MATLAB window.

   To open a new function file you use the `New->function` menuitem. To open an existing function file you use the `Open...` dialog.

b. We save the file as `f.m`; we have no choice if we used the name `f` when defining the function. The file should be saved in the default directory provided by the MATLAB function editor. This way MATLAB can find the function when it is requested.

   You do not need to close the function file editor after working on the file.

c. We reference the function by its name as we would any other function. That is, using parentheses and with arguments.

```
Define a function
using MATLAB's
editor



––––––––––––––
function y = f(x)
y=sin(x)./x;
```

```
Store on hardrive
asf.m
```

```
Using at command line



––––––––––––––
>> x = linspace(–pi,pi);
>> plot(x,f(x))
```
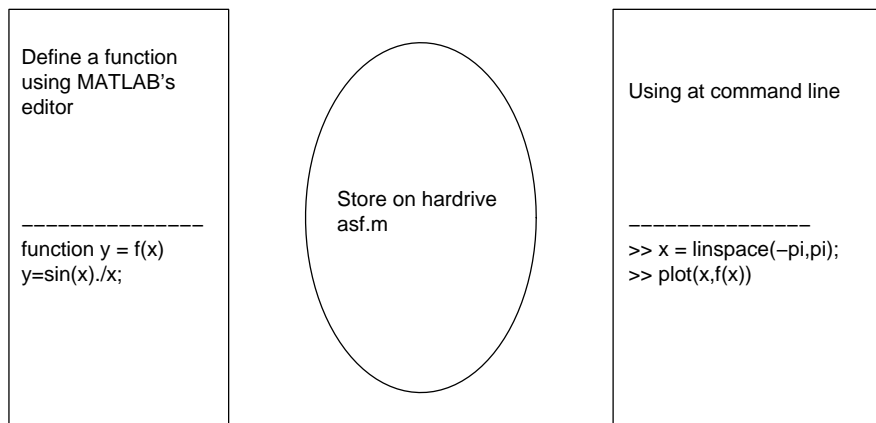
Figure 3: The three aspects to defining a function: the function template (with name), the naming of the file storing the function, using the function with a MATLAB session

Suppose we followed the instructions above and saved our template that defines $f(x) = \sin(x)/x$ in a file `f.m`. Then we can use our function as follows:

```
>> f(1)                       % call with a fixed value
ans = 0.8415
>> x=1;f(x)                   % call with an assigned variable
ans = 0.8415
```

```
>> x=[1,2,3];f(x)                    % call with a vector
ans =   0.8415     0.4546     0.0470
>> f                                 % needs a value as input to f.m
??? Input argument 'x' is undefined.
>> f.m(x)                            % the function is f(x) not f.m(x)
??? ... stuff skipped ...
```

Exercise 6:
We wish to find the right limit of the function $f(x) = x^{-1}e^{-1/x}$ as x goes to 0. (Notice it is indeterminate of the form $\infty \cdot 0$).

a. Write a function file to evaluate this function. Call the function f.m
(14)

b. Plot the function $f(x)$ over the interval $[0, 1]$ with these commands (using the definition of f)

```
>> x = linspace(0,1)
>> plot(x,f(x))
```

Use your graph to estimate the **right** limit as x goes to 0.
(15) Answer: _____

Exercise 7:
Write a function file to evaluate the function $g(x) = x^x$, even if x is a vector of numbers. Use the name g for this function so that both f and g may be referred to below.

What is your function template? (16)

Functions allow us to simplify certain calculations. For instance, composition of functions is easily handled by the computer, although it can be difficult by hand. For instance, what is $g(f(x))$ with the f and g from the previous examples? It is

$$g(f(x)) = (x^{-1}e^{-1/x})^{x^{-1}e^{-1/x}},$$

which may simplify, but we'll leave it like this to emphasize it need not be simple looking.

However, once functions are defined in MATLAB, we can perform composition quite easily. Just use `g(f(x))` in any commands.

Exercise 8:

a. Find the following limit using composition of functions:

$$\lim_{x \to 0+} g(f(x))$$

What is the value of the limit?
(17) Answer: _____

b. There are many consequences of the definition of limits that make finding limits easier. For instance, the limit of a sum is the sum of the limits (provided all the limits exist). Another is the limit of compositions can be expressed in terms of the limits. That is, if

$$\lim_{x \to c} f(x) = L, \quad \text{and} \lim_{x \to L} g(x) = M,$$

then

$$\lim_{x \to c} g(f(x)) = M.$$

Verify that this is the case with f and g defined as in the last problem. Is it true?
(18) Circle one: 1. yes **2.** no

Exercise 9:
The derivative of a function $g(x)$ at x is given by the following limit

The derivative at $x = \lim_{h \to 0} \dfrac{g(x+h) - g(x)}{h}$

Let $g(x) = x^x$. This should be defined in a function file called g. If not, do so now.

a. Find the derivative of $g(x)$ at $x = 1$ by investigating the above limit. Use the expression

```
>> y = (g(1+h) -g(1)) ./h
```

rather than doing the composition and secant line expression directly from the function definition. What is the value of the limit?
(19) Answer: _____

b. Now we try to find the right limit of the derivative formula when $x = 0$. As $g(0)$ is not defined, we replace it with it's limit of 1.

What do you find when investigating

$$\lim_{h \to 0+} \frac{g(h) - 1}{h}?$$

(20)

4   Function specifics

MATLAB's mechanism for defining functions is by the creation of "function" files. More generally, a function file is one of MATLAB's ways of storing information to a disc drive (others are `save` and `load`). These files are saved with the extension `.m` (for example `function.m`). They come in two flavors,

- **MATLAB script files:** these save a fixed sequence of commands.

- **MATLAB function files:** these save functions. A function takes some **input** (which is in a domain) and after evaluating some commands, returns the **output** (in its range).

Both are similar: they are both text files stored with the special suffix ".m".

Script files Script files are simply a way to store commands in a file that will get executed all at once. They are a convenient way to do work if it involves many lines of evaluation, as errors can be easily fixed.

To use a script file you need to

a. Create a file (say `script.m`) with the commands you want to execute.

b. Save this in the correct place, or tell MATLAB where it is by using the `cd` command to change to the directory the file is in. The following will cd to the "a" drive and look for files there.

```
cd a:
```

Alternatively, you can tell MATLAB where your file is by setting the `path`. This is done with a command like the following which adds the "a" drive to the path.

```
>> path(path,'a:\')
```

c. Execute the commands stored in the script file by calling the name from the command line without the `.m` extension. For example, if the file is called `script.m`, then the command line

```
>> script
```

will run the commands.

Function files Function files are very similar to the above, except they let you pass in values and return values. The simplest instance would be to mimic a mathematical function such as $f(x) = \sin^2(x)$.

To make a function file for this we would create a file `f.m` with these contents

```
function y = f(x)
y = sin(x).^2;
```

We save the file. If we put it in the correct place, running this file is easy, just like a built-in MATLAB function such as `sin(x)`. That is, we use parentheses and numbers. For example

```
>> f(2)
```

5   Functions can save work. Really!

Here is a sample function which can actually save time! To graph a function in MATLAB is somewhat tedious. You need to define x usually with linspace, then define y, and then define the plot commands. Here is a function which will do this all for you. Save this function as `plotf.m`

```
function y = plotf(a,b,n)
% plotf: plots f over [a,b]. f is  function file previously defined
if (nargin <3) n=100; end
x = linspace(a,b,n);plot(x,f(x))
```

Now you can easily plot and replot a function.

6   Things to watch out for with function files

There are several things that can go wrong with function files. If you are having trouble you might try looking here.

Documenting function files The `help` command is very useful. To add a help string to your file is easy. It will be the first block of comments immediately after the function keyword. For example if `f.m` contained the file

```
function y = f(x)
% this is a function
y=x.^2;
```

then from the command window we would have

```
>> help f
   this is a function
```

Error messages when calling the file It is important to remember that script files and function files are called in different manners. If you have a file called `f.m` then if it is a script file it is called on the command line with

```
>> f
```

Whereas, if it is a function file, doing this will cause an error. Functions need to have arguments. You call them like

```
>> f(x)
```

where x is some previously defined number or list of numbers.

MATLAB can't find your function file MATLAB has a path that it uses to search for files. If your file isn't on the path, MATLAB won't know about it. To ensure that your file is on the path you can change to the current directory. For example, if the file is on a diskette, then the command `cd a:\` should suffice. Otherwise you can adjust the `path` variable with `>> path(path,'a:\')`.

MATLAB can find a file, but it doesn't look like mine When MATLAB looks along its path for a function or script file, it finds the first thing it can. In particular, if someone has a file with the exact same name as yours, but it appears first on the path, then that function file will be the one executed. Usually, the current directory is first on the path, and so the `cd` command above will usually suffice.

MATLAB can't find your script file The syntax to call a script file is identical to that to view the contents of a variable. This is not a problem unless you have a variable with the same definition as your script file. In this case the variable's contents are given and not your script file.

MATLAB finds a variable before a function If you happen to have a variable with the same name as your function, then the variable will be returned. For example, suppose you have a function file `f.m` containing

```
function y = f(x)
% this finds x e^(1/x)
y = x.* exp(1 ./ x);
```

Then we would have the following session

```
>> f(2)                         % call the function
ans = 3.2974
>> f = 1                        % set a variable f
f =   1
>> f                            % view the variable
f = 1
>> f(2)                         % it tries the variable not the function
???  Index exceeds matrix dimensions.
>> clear(''f'')                     % clear the variable. Oops
??? clear(''f       |
Use single quote character instead of double quote or backward quote.
>> clear('f')                   % Okay, single quotes.
>> f(2)                         % Now it works again
ans = 3.2974
```

The College of Staten Island
Department of Mathematics

Approximate First and Second Derivatives

1   Introduction

The derivative of a function f at x is defined in terms of a limit, which may be written as

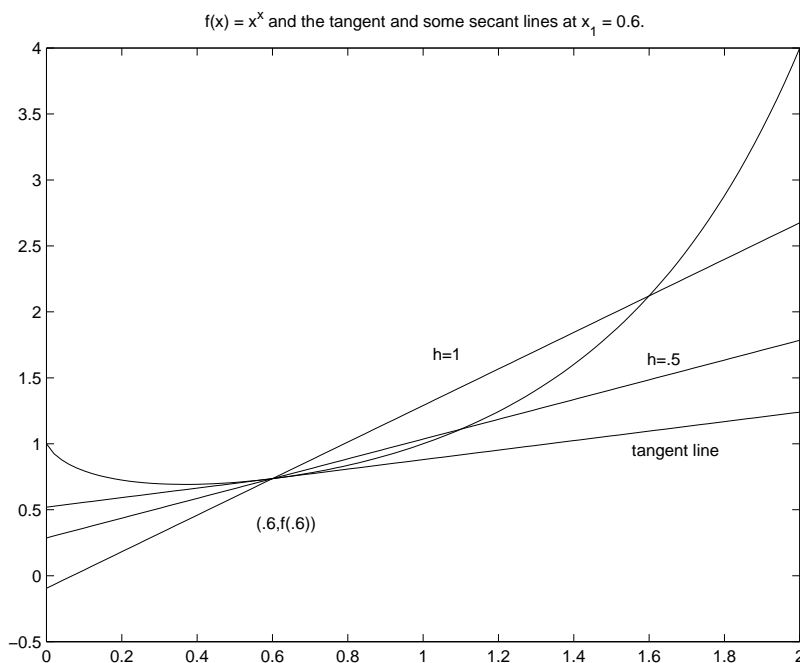$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

Figure 1: Graph of the function $f(x) = x^x$ over $[0, 2]$. Layered on top are the tangent line at $x = 0.6$ and two secant lines corresponding to $h = 1.0$ and $h = 0.5$. As h approaches 0, the slope of the secant line approaches the slope of the tangent line.

The expression that we take the limit of,

$$\frac{f(x+h) - f(x)}{h}$$

is interpreted as the slope of the secant line that goes through the points $(x, f(x))$ and $(x+h, f(x+h))$. Then the derivative is interpreted as a slope, in this case of the tangent line, defined as the limit of the slopes of the secant lines.

We've seen that MATLAB can be used to investigate limits. Although, MATLAB doesn't actually find the limit, it suggests an answer by being close to the true answer. In this same way, MATLAB can help us find the value of the derivative of $f(x)$ at a single point, and as a function of x (many points at once.)

Figure 1 shows that graph of $f(x) = x^x$ over the interval $[0, 2]$ along with three lines which go through the point $(0.6, f(0..6))$. The tangent line is drawn to have slope given by $f'(x)$ evaluated at 0.6. The secant lines are drawn to have slope which depends on a specified value of h. For instance, when $h = 1$ the slope is

$$\frac{f(0.6 + 1) - f(0.6)}{1} = \frac{2.1213 - 0.7360}{1} = 1.3852.$$

Clearly this is not close to the derivative, since the slope of this secant line is not close to the slope of the tangent line. We don't expect it to be, as h is not close to 0. Taking values of h getting closer to 0 allows us to approximate the derivative at 0.6:

```
>> c = 0.6; h = [1.0 0.1 0.01 0.001 0.0001];
>> y = ( (c+h).^(c+h) - c^c ) ./h;
>> [h; y]'
ans =
   1.0000   1.3852
   0.1000   0.4303
   0.0100   0.3671
   0.0010   0.3607
   0.0001   0.3601
```

We'll take the value $f'(0.6) = 0.3601$ as the approximate derivative. This is about as good as we can get before learning how to find the actual derivative algebraically.

The same approach to finding limits can be done for many x values at once. This allows us to graph an approximation to the actual derivative of a function f. As well, we'll see how to plot an approximate second derivative. We emphasize that these are approximate, as we will use a fixed value of h, which although small, is still not "going to" 0.

2   The derivative at a point

Exercise 1:
Let $f(x) = \sin(x^2)$. We wish to find the derivative of f when $x = \pi/4$.

   a. Make a function file storing the function $f(x) = \sin(x^2)$ with name `f.m`. What are the contents of the file:
      (1)

b. Take $h = 0.1$ and find the slope of the secant line at $(x, f(x))$:

$$\frac{f(\frac{\pi}{4} + h) - f(\frac{\pi}{4})}{h}.$$

(2) Answer: _____

c. Take smaller values of h to find a better approximation to $f'(\pi/4)$ (Four decimal points of accuracy is good.)
   (3) Answer: _____

Exercise 2:
We will see how the secant line approaches the tangent line for the function $f(x) = \sin(x)$ at the point $x_1 = \pi/4$. Do this by following this sequence of steps. When you are done you will upload your graph to show your work.

A secant line is just a line drawn between two points on the graph of $f(x)$, where one of the points is the point of interest like $x_1 = \pi/4$ in this example. The second point can be called x_2, but we prefer to emphasize its relationship to x by writing $x_2 = x + h$ where h then is the distance between the two points.

a. Make a function file storing the function $f(x) = \sin(x)$ with name f.m. What are the contents of the file:
   (4)

b. Graph $f(x)$ over the interval $[0, \pi]$.

c. Let $h = \pi/2$. On your graph, add the secant line connecting the points $x_1 = \pi/4$ and $x_2 = x_1 + h = 3\pi/4$.

   The following commands will help you plot such a line.

```
>> x1=pi/4;
>> h = pi/2;  x2 = x1+h; m = (f(x1+h)-f(x1)) / h ;
>> plot(x, f(x1) + m*(x-x1))            % point-slope form of line
>> plot(x2,f(x2),'*')                   % mark the point
```

   What is the slope of this secant line?
   (5) Answer: _____

d. For $h = 0.1$, and $h = 0.001$ draw the secant line as above. Label the different secant lines with the value of h.

e. The tangent line at $x_1 = \pi/4$ has slope $\sqrt{2}/2$. Plot the tangent line on the same graph in a different color and label it.

f. Submit the graph of Example 1.

   (6) **Attach your graph to the worksheet.**

g. Explain carefully the relationship between h and slope of the secant lines for h. In particular, what is the limit of these slopes as h gets closer to 0?

   (7)

3   Derivative of a Function

We've seen that the derivative of $f(x)$ at c is approximately given by $(f(c+h)-f(c))/h$ for small values of h. To find the derivative of a function for many x values at once, we simply fix a single small value of h, and use many values for c. We call this approximate derivative the difference quotient:

$$\text{difference quotient} = \frac{f(x+h) - f(x)}{h} \qquad \text{small } h.$$

Example 1:
(Graphing the difference quotient)
Let us create a plot of the derivative of $f(x) = \cos x$ over the interval $[0, 2\pi]$ based on the numerical evaluation of its difference quotient. We shall use $h = 0.01$ in evaluating the difference quotient. The following MATLAB commands will plot both the function and its **approximate** derivative together over the interval $[0, 2\pi]$.

```
>> h=0.01;                    % a small value for h
>> x=linspace(0,2*pi);        % lots of x values
>> difquo=(cos(x+h)-cos(x))./h; % difference quotient. Or use a function file
>> plot(x,cos(x),'r',x,difquo,'b'),grid % plot cos in red and difquo in blue
```

Exercise 3:

Have a close look at the resulting graphs of Example 2. Identify which is the graph for `cos(x)` and which is the graph for `difquo`.

    a. Does the graph of `difquo` resemble the graph of a function that you are familiar with?

       (8) Circle one: 1. $\sin x$ **2.** $-\sin x$ **3.** $\cos x$ **4.** $-\cos x$

    b. Based on your answer to part 1, what do you think is the derivative of $y = \cos x$?

       (9) Circle one: 1. $\sin x$ **2.** $\cos x$ **3.** $-\cos x$ **4.** $-\sin x$

Exercise 4:

It is known that the derivative of the function $f(x) = x^{3/2} + 5$ is given by $f'(x) = 3/2\sqrt{x}$. We will verify that the difference quotient converges to this function as h goes to 0 over the interval $[0, 2]$

    a. Write a function file storing the definition of $f(x)$. List its contents here:
       (10)

    b. For $h = 0.1$ make a graph of the difference quotient and the derivative. What is the largest difference between the two functions?
       (11) Answer: _____

    c. On the same graph, plot the difference quotient for $h = 0.001$. What is the maximum difference now between the difference quotient and the derivative function?
       (12) Answer: _____

Exercise 5:
Numerical differentiation of exponential function

You may not know the derivative of $f(x) = e^x$ and $g(x) = \ln(x)$ at this point. Let's use the difference quotient to figure out the derivative of e^x.

a. Use MATLAB to create a continuous plot of $f(x) = e^x$ over the interval $-2 \leq x \leq 2$. Based on this graph, which of the following should be true about the derivative.
(13) Circle all that apply:
1. $f(x)$ is always increasing so $f'(x) > 0$.
2. Tangent lines for $f(x)$ are never 'flat' so there are no critical points.
3. $f(x)$ increases faster and so $f'(x)$ should be increasing

b. On top of the graph of $f(x)$, create a plot of the difference quotient of $f(x)$ using $h = 0.01$ over the same interval $[-2, 2]$ in a different color. Turn in your graph.
(14) **Attach your graph to the worksheet.**

c. Based on the two plots, what do you think the derivative of $f(x) = e^x$ is.
(15) Circle one: 1. x^e **2.** $\log x$ **3.** x^2 **4.** e^x

4 The approximate second derivative

The second derivative is approximated in a similar way as the first derivative is approximated. A common approximation is given by

$$\texttt{difdifquo} = \frac{f(x+h) - 2f(x) + f(x-h)}{h^2}.$$

It can be checked that as h approaches 0, the `difdifquo` (for lack of a better name) converges to $f''(x)$.

Using a function file makes it easier to find the approximate second derivative. For instance, the following example will plot a function $f(x)$ and its approximate second derivative over the interval $[0, 1]$. It just needs a function file defining the function `f`.

```
>> x = linspace(0,1);         % adjust for your problem
>> h = 0.01;                  % a small enough number?
>> plot(x,f(x),x,(f(x+h) - 2*f(x) + f(x-h))/h^2,'g'),grid
```

That's it.

Exercise 6:

We wish to graph the function $f(x) = e^{x/3} \sin(\pi x)$.

a. Write a function file that evaluates this function. Call your file f.m.
   (16)

b. Make a plot of $f(x)$ over the interval $[0, 3]$.

c. Using $h = 0.01$ compute the difference quotient for x in the interval $[0, 3]$. Write down the commands to find the difference quotient and add its graph to the graph of $f(x)$.
   (17)

d. Again using $h = 0.01$, find the approximate second derivative (difdifquo) and add its plot to your plot of the function. What are the commands you used to find difdifquo?
   (18)

e. Use an arrow to label any zeroes of the approximate first derivative on your graph. What can you say about the graph of $f(x)$ near these x values?
   (19)

f. Submit your three graphs in one figure with the requested annotations.
   (20) **Attach your graph to the worksheet.**

MTH229

<div align="center">

The College of Staten Island
Department of Mathematics

Critical and Inflection Points

</div>

1   Introduction

This project explores the important relationships between a function, $f(x)$, and its first and second derivatives, $f'(x)$ and $f''(x)$. For instance, we know:

- $f'(x)$ is positive on an open interval is equivalent to $f(x)$ is increasing on the interval.

- $f'(x)$ is increasing on an open interval is equivalent to $f(x)$ being concave up on the interval.

- $f''(x)$ is positive on an open interval is equivalent to $f(x)$ being concave up on the interval.

   These facts lead to the first and second derivative tests, which allow us to check if a critical point is indeed a relative maximum or a relative minimum. Recall, a critical point, c, is a value in the domain of $f(x)$ where $f'(c) = 0$ or $f'(x)$ is undefined at $x = c$.

2   The first derivative

We investigate the function

$$f(x) = 4x^4 - 12x^3 + 9x^2.$$

```
>> x=linspace(-3,3);
>> y=4*x.^4 - 12*x.^3 + 9*x.^2;
>> plot(x,y), grid
```

   A plot over the interval $[-3, 3]$ reveals an apparent "flat section" with no visible relative extrema. To produce a plot that reveals the true structure of the graph, we replot over the interval $[-1, 2]$:

```
>> x=linspace(-1,2);
>> y=4*x.^4 - 12*x.^3 + 9*x.^2;
>> plot(x,y), grid
```

We see two relative minima, one relative maximum and two points of inflection. Next, we plot the derivative function, $f'(x)$, which we name yp for short, on the same graph as $f(x)$ in red.

```
>> hold on
>> yp=16*x.^3-36*x.^2+18*x;
>> plot(x,yp,'r')
```

We know that the zoom method works poorly to find the exact value of a relative maximum or minimum, as it is hard to find out exactly where the true graph is flat from the approximation that MATLAB produces. However, at a relative extrema, the function has a critical point, so in many cases the derivative is zero. We can then zoom in on the graph of $f'(x)$ to investigate the relative extrema.

Exercise 1:

a. Find the zeros of $f'(x)$ correct to two decimal places.
   (1) Answer: _____

b. How do the zeros of $f'(x)$ relate to $f(x)$?

   (2) Circle one:
   1. They do not relate to $f(x)$
   2. they're the values for which f(x) has its critical points
   3. they're the values for which f(x) = 0

c. From its graph, find the intervals on which $f'(x)$ is negative?

   (3) Circle one:
   1. (0,0.75) , (1.5,∞)
   2. (-∞,0) , (0.75,1.5)
   3. (-∞,0) , (0.74,1.2)
   4. not listed

d. What can you determine about $f(x)$ on these intervals?

   (4) Circle one:
   1. $f(x)$ increases where $f'(x) > 0$ and decreases where $f'(x) < 0$
   2. $f(x)$ increases where $f'(x) < 0$ and decreases where $f'(x) > 0$
   3. $f'(x)$ increases where $f(x) > 0$ and decreases where $f(x) < 0$
   4. none of the above

e. Use the zeros of $f'(x)$ to find the coordinates of the relative maximum of $f(x)$ correct to two decimal places. (Note: After finding the proper value of x let the computer calculate the corresponding value of y. First enter the value of x. Recall y with the up-arrow key and enter. Then type y and press the enter key).

f. The coordinates of the relative maximum are:
   (5) Circle one:
   1. (0.63,1.76) and (1.27,4,78)
   2. (0.55,2,43)
   3. (0.75,1.27)
   4. not listed

g. Find the coordinates (both x and y), correct to two decimal places, of the absolute maximum and the absolute minimum of $f(x)$ on the interval [0,2].

h. The absolute maximum is:

   (6) Circle one:
   1. (2,4) endpoint
   2. (2,4) critical number
   3. (1.5,1.2) critical number
   4. not listed

i. The absolute minimum is:
   (7) Circle one:
   1. (1,0) critical number
   2. (1.2,.3) critical number
   3. (0,0) and (1.5,0) both critical numbers
   4. not listed

3   The second derivative

Exercise 2:

To find the points of inflection, we examine the zeros of $f''(x)$, the second derivative. We use the name ypp for this.

```
>> hold on
>> ypp= 48*x.^2 - 72*x + 18;
>> plot(x,ypp,'g')
```

   a. On what intervals is $f''(x)$ positive?
      (8) Circle one:
      1. (-3,1.83)
      2. (0.3170,1.1830) and (1.1830,∞)
      3. (-∞,1.1830)
      4. (-∞,0.3170) and (1.1830,∞)

   b. On what intervals is $f''(x)$ negative?
      (9) Circle one:
      1. (0.3170,1.1830)
      2. (0.3170,1.1830) and (1.1830,∞)
      3. (-∞,1.1830)
      4. (-∞,0.3170) and (1.1830,∞)

   c. What is the relationship between the sign of the function $f''(x)$ and the concavity of $f(x)$.
      (10) Circle all that apply:
      1. $f''(x) > 0$ on [a,b] implies that $f(x)$ is concave down on $[a, b]$
      2. $f''(x) < 0$ on $[a, b]$ implies that $f(x)$ is concave up on $[a, b]$
      3. $f''(x) < 0$ on $[a, b]$ implies that $f(x)$ is concave down on $[a, b]$
      4. $f''(x) > 0$ on $[a, b]$ implies that $f(x)$ is concave up on $[a, b]$

   d. Find the coordinates of the points of inflection of $f(x)$ correct to two decimal places by finding the zeros of $f''(x)$ graphically. Give both coordinates of each inflection point. Again, use $y = f(x)$ to calculate the y-coordinate of the inflection point.
      The coordinates are: (Be careful.)
      (11) Circle one:
      1. (0.31,0.52) and (1.14,0.44)
      2. (0.29,0.52) and (1.78,0.44)
      3. (0.31,0.50) and (1.14,0.44)
      4. (0.32,0.56) and (1.18,0.56)

e. Find $f''(x)$ and solve for its zeros using the quadratic equation. Find the x-coordinates of the inflection points and compare to the results found graphically.
The coordinates are:
(12) Circle one:
1. $2 - \sqrt{2}$ and $2 + \sqrt{2}$
2. $(3 + \sqrt{3})/4$ and $3/4$
3. $(3 - \sqrt{3})/4$ and $(3 + \sqrt{3})/4$
4. $(3 - \sqrt{3})/4$ and π

f. Attach a graph of $f(x)$, $f'(x)$ and $f''(x)$. Please identify the three functions on your graph.
(13) **Attach your graph to the worksheet.**

Exercise 3:
This exercise uses information about the derivative of a function to infer properties of the unknown function. While it is true that we can not completely reconstruct $f(x)$ from $f'(x)$ without some additional detail, we can say characterize for $f(x)$ its relative extrema and concavity.

Let the derivative of a function $f(x)$ be given by $f'(x) = x^3 - 7x^2 + 14$. We'll investigate the behavior of $f(x)$ given this information.

Graph both $f'(x)$ and $f''(x)$ on the interval $[-4, 8]$ with a grid using a different color for each. Add a title with your name, and label the graphs in some manner.
We will use both graphs to answer the following questions about the unknown function $f(x)$.

(**Note:** Since in this exercise $f'(x)$ and $f''(x)$ are polynomials, instead of zooming, you could use the MATLAB `roots` command to find their zeros accurately. For example to find the roots of the polynomial $x^2 - x - 1$ you use the representation `[1 -1 1]` for the polynomial and the command `roots([1 -1 -1])` to find the roots.)

a. On what subinterval(s) is $f(x)$ increasing?
(**14**) **Circle one:**
1. (-1.29,1.61) and (6.69,∞)
2. (-∞,0) and (4.67,∞)
3. (-∞,0)
4. (-∞,0) and (6.69,∞)

b. Which of the following conditions did you use to find the subinterval(s) on which $f(x)$ increases?

   (15) Circle one:
   1. $f'(x) > 0$ on these subintervals
   2. $f''(x) > 0$ on these subintervals
   3. $f'(x) < 0$ on these subintervals
   4. $f'(x)$ is increasing on these intervals

c. Find the x-coordinates of all relative minima of $f(x)$.

   (16) Circle one:
   1. $x = 4.56$
   2. $x = -1.29$ and $x = 6.69$
   3. $x = 4.56$ and $x = 0$
   4. $x = -1.29$ and $x = 1.61$

d. On what subinterval(s) is $f(x)$ concave up?

   (17) Circle one:
   1. $(-\infty, 0)$ and $(4.67, \infty)$
   2. $(1.166, \infty)$
   3. $(-\infty, 1.167)$
   4. it's always concave down

e. Which of the following conditions did you use to find the subinterval(s) on which $f(x)$ is concave up?

   (18) Circle one:
   1. $f'(x) > 0$ on these subintervals
   2. $f''(x) > 0$ on these subintervals
   3. $f'(x) < 0$ on these subintervals
   4. this can only be known by examining the graph

f. Find the x-coordinates of all points of inflection of $f(x)$.

   (19) Circle one:
   1. $x = 4.56$ **2.** $x = 4.65$ **3.** $x = 14/3$ and $x = 0$ **4.** not listed

g. Submit your graph of $f'(x)$ and $f''(x)$.

   (20) **Attach your graph to the worksheet.**

Exercise 4:

    a. Use MATLAB to graph $f(x) = x - \sin(x)$ on the interval $[0, 4\pi]$. Use "hold on" to graph $f'(x)$ and $f''(x)$ on the same plot. Use MATLAB to label the three functions.

       (21) Attach your graph to the worksheet.

    b. Does $f(x)$ have any relative extrema on this interval? (Recall a relative extrema requires an open interval.)

       (22) Circle one:
       1. at $x = 0, 2\pi, 4\pi$
       2. at $x = \pi/2 \ x = 5\pi/2$
       3. none
       4. at $x = \pi/2 \ x = 2\pi$

    c. Give an explanation for your previous answer.
       (23) Circle one:
       1. these are the values where $f'(x) = 1 - \cos(x) = 0$
       2. these are the values where $f'(x) = 1 - \cos(x) > 0$
       3. these are the values where $f''(x) = \sin(x) = 0$
       4. There are no values where $f'(x)$ changes sign

    d. Identify all points of inflection. in $(0, 4\pi)$ (Do not include the endpoints.)
       (26) Circle one:
       1. (π, π)
       2. $(\pi, \pi), (2\pi, 2\pi), (3\pi, 3\pi)$
       3. $(\pi, \pi), (3\pi, 3\pi), (4\pi, 4\pi)$
       4. $(\pi, \pi), (2\pi, 2\pi), (3\pi, 3\pi), (4\pi, 4\pi)$

Exercise 5:

The concentration f of a certain medicine in the bloodstream t hours after injection into muscle tissue is modeled by:

$$f(t) = \frac{3t^2 + 1}{50 + t^3}, \quad t \geq 0$$

Use the graphical capability of MATLAB to investigate the model.

a. Make a graph of the concentration, $f(t)$ for $t \geq 0$. (You need to decide how large t should be to answer the questions below.) On the same graph plot $f'(t)$ and $f''(t)$ or the approximate first and second derivatives, `difquo` and `difdifquo` of a previous project. Label the graphs. (It is easier to plot `difquo` and `difdifquo` as the derivative gets messy. For example if `t` has already been defined and you created a function file `f.m` then

```
>> h=.01;plot(t,(f(t+h)-f(t))/h, t, (f(t+h)-2*f(t)+f(t-h))/h^2))
```

will plot them.)

(27) **Attach your graph to the worksheet.**

b. When will there be maximum concentration?

(28) **Answer:** _____

c. How much is the maximum concentration?

(29) **Answer:** _____

d. When will the concentration dip below a level of 0.1?

(30) **Circle one:** **1.** $t = 30$ **2.** $t = 40$ **3.** $t = 50$ **4.** $t = 55$

e. Estimate graphically where the concentration function changes concavity?

(31) **Answer:** _____

f. In this model, is the concentration ever zero?

(32) **Circle one:** **1.** yes **2.** no

4   summary

In this project, we have used the graphical capability of MATLAB to find important properties of a function $f(x)$. Specifically, the relationship between $f'(x)$ and the increasing/decreasing properties of $f(x)$, and that between $f''(x)$ and the curvature of $f(x)$. Then you have used these ideas to accurately determine the coordinates of the relative extrema and the points of inflection of $f(x)$.

MTH229

The College of Staten Island
Department of Mathematics

Newton's method

1 Introduction

The quadratic formula is one of the most beautiful in mathematics. For any quadratic polynomial, it allows us to say without fuss exactly what its zeroes are. However, in many instances in mathematics, an answer is not given so directly in terms of a formula, such as the quadratic formula, but rather in terms of an algorithm that produces the answer.
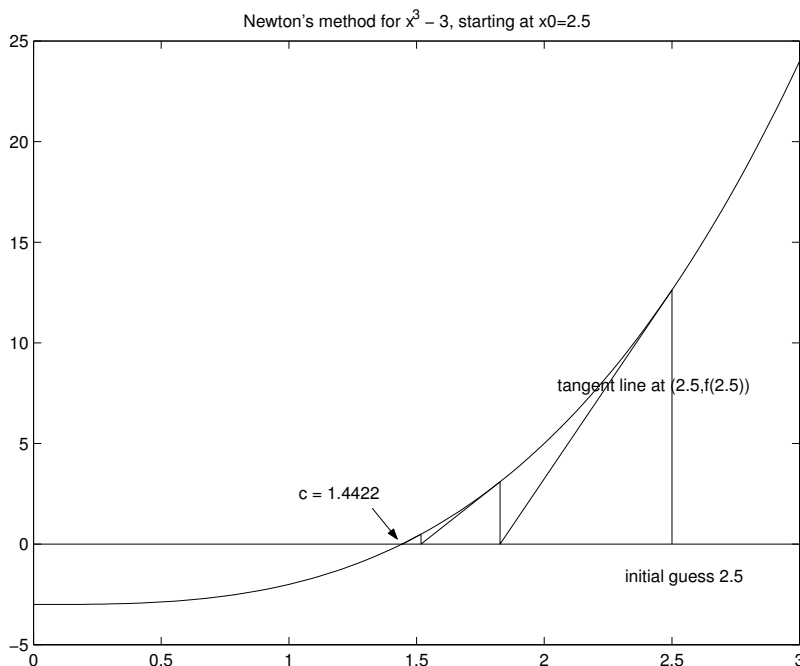
Figure 1: Illustration of a few iterations of Newton's method for finding zero of $x^3 - 3$.

Newton's method is an iterative algorithm that is used to find the zeroes of a function when other direct formulas are not known. We've seen that the zeroes of the equation, $f(x) = 0$, can be found by zooming in on the graph of $f(x)$. However, this has limitations if we need a lot of accuracy. In this case, Newton's method can usually provide the accuracy.

Newton's method works as follows: we have a zero, say $\tilde{x}$, that we are trying to find (that is, $f(\tilde{x}) = 0$). We might have an initial guess for $\tilde{x}$ that we assume is close, but we know is not likely to be the zero. Call this initial guess x_0. Starting with an initial guess, Newton's method produces a second one x_1 which should be closer to $\tilde{x}$. From x_1 the method produces x_2, and so on, and so

on, giving a sequence $x_0, x_1, x_2, \ldots, x_n, \ldots$. Figure 1 shows a few iterations of Newton's method applied to the function $x^3 - 3$ with an initial point $x_0 = 2.5$.

We don't expect any of the x_i to be equal to $\tilde{x}$, only that in the limit the x_n converge to $\tilde{x}$. As we did with limits, we will settle for values that seem close to the desired value. We would like to stop at some n so that x_n is within some tolerance of $\tilde{x}$. However, we don't know $\tilde{x}$, so instead we stop when the difference between subsequent guesses, x_n and x_{n+1}, is small, where small is determined ahead of time. This may be something like they agree to 4 decimal places, or for more accuracy, may depend on the machine precision. (This is given by the MATLAB variable eps.)

Now, to describe Newton's method, we need to say how it takes one guess, and produces another. Call the starting guess x_n and the new guess x_{n+1}. Figure 2 shows the tangent line at $(x_n, f(x_n))$ and defines x_{n+1} as the intersection of this line with the x-axis. That's it. From x_n go vertically to the graph and then back to the x-axis along the tangent line. Repeat until this procedure fails to make a difference between the two guesses.
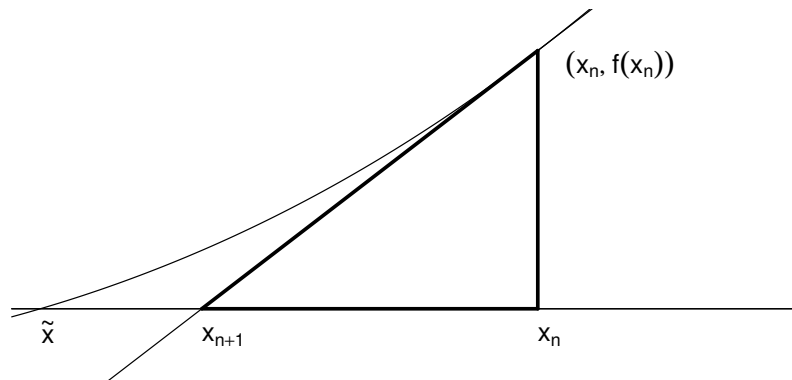

Figure 2: Illustration of Newton's method showing how to find x_{n+1} from x_n. The darkened triangle relates known values for the slope with the rise and run.

Algebraically, the tangent line forms a triangle with slope given by $f'(x_n)$, rise given by $f(x_n)$, and run given by $x_n - x_{n+1}$. As the slope is the rise over the run we have

$$f'(x_n) = \frac{f(x_n)}{x_n - x_{n+1}}$$

Solving for the new value, x_{n+1}, we get Newton's method:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

Example 1:

We illustrate Newton's method by determining a numeric value for the solution to $x^3 = 3$ or $f(x) = 0$ with $f(x) = x^3 - 3$. We make an initial guess for the root $x_0 = 5$.

Newton's method says that

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = x_0 - \frac{(x_0^3 - 3)}{3x_0^2},$$

using the fact that $f'(x) = 3x^2$ by the power rule. How do we code this for MATLAB?

For iterative schemes like Newton's method, it pays to exploit the difference between the equals sign in mathematics and its role in a computer language. In MATLAB, the equals sign means assignment. So the statement

```
>> x = x - (x^3-3)/(3*x^2)
```

takes the current value of x and replaces it according to the algorithm. So after evaluating this line, the variable x contains the equivalent of x_{n+1}.

We use this trick below to implement Newton's method for this problem.

```
>> x = 5
x = 5                              % x0 = 5
>> x = x - (x^3 - 3)/(3*x^2)
x = 3.3733                         % x1 = 3.3733
>> x = x - (x^3 - 3)/(3*x^2)
x = 2.3368                         % x2 = 2.3368
>> x = x - (x^3 - 3)/(3*x^2)
x = 1.7410                         % x3 = 1.7410
>> x = x - (x^3 - 3)/(3*x^2)
x = 1.4906                         % x4 = 1.4906
>> x = x - (x^3 - 3)/(3*x^2)
x = 1.4438                         % x5 = 1.4438
>> x = x - (x^3 - 3)/(3*x^2)
x = 1.4423                         % x6 = 1.4423
>> x = x - (x^3 - 3)/(3*x^2)
x = 1.4422                         % x7 = 1.4422
>> x = x - (x^3 - 3)/(3*x^2)
x = 1.4422                         % x8 = 1.4422
```

We stopped after 8 iterations, as x_7 and x_8 are identical to four decimal points. This is a tolerance of 10^{-4}. Had we turned on `format long`, we could get more accuracy.

How accurate is our solution? We evaluate the function at the currently stored value of x (x_8) to see:

```
>> x^3 - 3
ans = -4.4409e-16
```

This is basically the machine tolerance, so we would be hard pressed to do better. As such, we will refer to this value as $\tilde{x}$, even though it isn't technically correct.

Exercise 1:
Evaluate $f(x)$ at the value for x_8 displayed in the example, x=1.4422. Is $|f(1.4422)| < 10^{-5}$?
(1) Circle one: 1. yes **2.** no

In the following exercises, you will be asked to apply Newton's method. You may want to use function files to store the functions $f(x)$ and $f'(x)$. If these are called f.m and fp.m, respectively then the following command will perform Newton's method:

```
>> x = x - f(x)/fp(x)            % only if f.m, fp.m defined properly
```

Exercise 2:
Let's use Newton's method to find the value of $\sqrt{2}$. (This is a zero of the equation $f(x) = x^2 - 2 = 0$.)

- Start with an initial guess of $x_0 = 3$ and use Newton's method to approximate the zero, $\tilde{x}$. What is the answer?
  (3) Answer: _____

- How many iterations did it take?
  (4) Answer: _____

- What is the value of $f(\tilde{x})$?
  (5) Answer: _____

- Print out a graph of $f(x) = x^2 - 2$ on the interval $(-4, 4)$. Using a straight edge (another piece of paper or a ruler) graphically do Newton's method on your printout starting with a value of $x_0 = 3$ and $x_0 = -1$. Do you get the same root each time?
  (6) Circle one: 1. yes **2.** no

Of course, we could have just found the last answer with the command sqrt(2). Here is another exercise which isn't quite so easy to solve.

Exercise 3:
Apply Newton's method to find a root of $f(x) = x^3 + 2x^2 - 30x - 5$.

    a. First, plot the function $f(x) = x^3 + 2x^2 - 30x - 5$ over the interval $(-5, 5)$. How many times does the graph cross the x-axis? (Why can't it be more than 3?)
        (7) Circle one: 1. 1 **2.** 2 **3.** 3 **4.** 4

    b. Find the largest root using Newton's method starting at $x_0 = 4$. It should be clear that the largest root has a value between 4 and 5 from your graph. Zero in on this root. What value do you find? (use `format short`)
        (8) Answer: _____

    c. How many iterations did it take?
        (9) Answer: _____

    d. Check that your value of $f(\tilde{x})$ is close to 0. If it is not, you made a mistake.

(Of course you could check this answer with the `roots` command.)

Exercise 4:
Let $f(x) = e^x - x^4$. (Then $f'(x) = e^x - 4x^3$.) This function has three zeroes in the interval $[-1, 10]$.

    a. What zero is found from Newton's method when it is started with $x_0 = -1$?
        (10) Answer: _____

    b. What zero is found from Newton's method when it is started with $x_0 = 7$?
        (11) Answer: _____

    c. What zero is found from Newton's method when it is started with $x_0 = 8$?
        (12) Answer: _____

    d. What zero is found from Newton's method when it is started with $x_0 = 7.3$?
        (13) Answer: _____

2   Why does Newton's method work?

Why does Newton's method work? When does it fail to work? Some insight into these questions can be found in Figure 3. The figure shows one step in Newton's method. Starting at x_n, the point x_{n+1} is found from the tangent line at $(x_n, f(x_n))$. Notice that if we knew the secant line connecting $(\tilde{x}, 0)$ to $(x_n, f(x_n))$ we could find $\tilde{x}$ by following it to the x axis. However, we only know the tangent line. Yet, if we have reason to believe that the tangent line is a good approximation to the desired secant line, then following the tangent line to the new point x_{n+1} should leave us close to $\tilde{x}$. For this picture, we see it is closer than x_n–the value found by taking a vertical line to the x-axis.
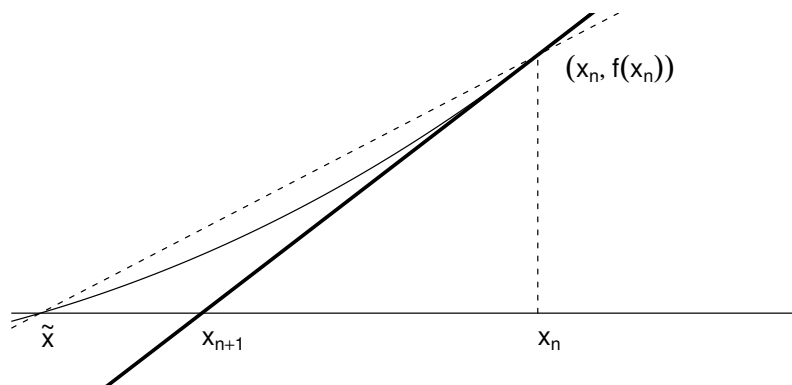
Figure 3: The tangent line at $(x_n, f(x_n))$ is in between the secant line connecting $\tilde{x}$ to the same point, and the vertical line connecting x_n. From here we see that the value of x_{n+1} is closer to $\tilde{x}$ than the value x_n.

Now we know that the secant line is a good approximation to the tangent line if the points are close, so if our guess in Newton's method is good, we should have the two lines being close, and consequently Newton's method should work well. However, if the tangent line is not close to the secant line, then all bets are off. There is no reason to believe that the method will converge on the zero that you are investigating. This is hinted at in the form of Newton's method, where we divide by $f'(x_n)$. Clearly if this value is 0 we will have problems as we are dividing by 0. Geometrically we can see this as well as the tangent line in this case will not intercept the x-axis, and so Newton's method will not produce a good point.

The following examples show what can happen if the initial point is not well chosen, or the function itself does not behave well for Newton's method.

This exercise illustrates that Newton's method may not find the answer near where you start, if the initial point is not well chosen.

Exercise 5:

Solve the equation

$$\sin(x) = x/4,$$

using Newton's method starting with $x_0 = 2\pi$. Figure 4 illustrates the algorithm graphically.
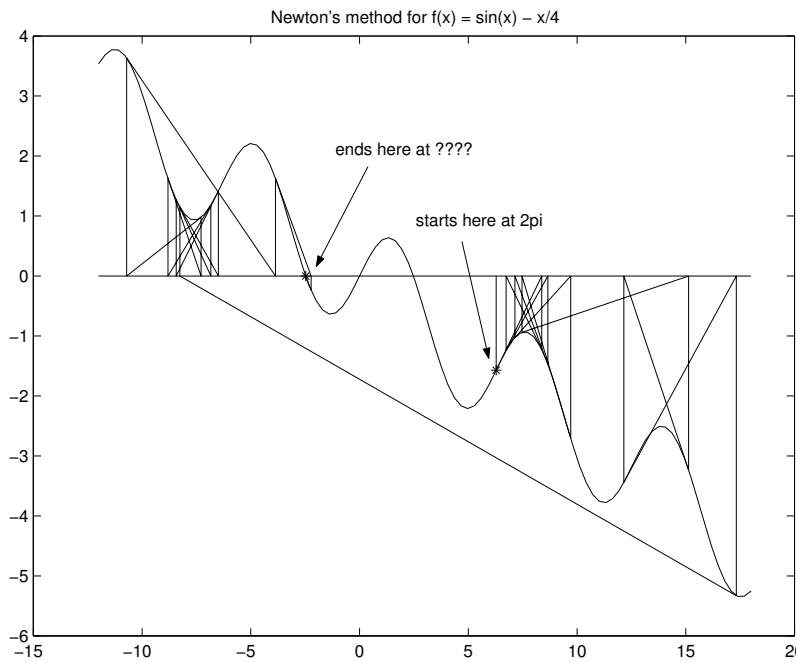

Figure 4: Example of Newton's method where convergence is not as expected due to poor choice of initial point.

   a. What is the value returned by Newton's method for the zero?
      (14) Answer: _____

   b. How many iterations did it take?
      (15) Answer: _____

    This example shows an example where Newton's method fails to converge for some starting values.

Exercise 6:

    a. Factor $x^3 - 5x$ to find the exact zeroes. How many are there?
        (16) Circle one: **1.** 0 **2.** 1 **3.** 2 **4.** 3

    b. What are they? (enter square roots as `sqrt(x)`)
        (17) Answer:

    c. Let the initial guess be 1. Use Newton's method to try and find a zero. What happens? (be specific)
        (18)

    d. Does this same behavior occur with $x_0 = 2$? What answer does Newton's method give for $\tilde{x}$ in this case?
        (19) Answer: _____

To fully understand what is going on with this example, trace a few iterations of Newton's method with your finger on your graph.

   This example also shows an example where Newton's method fails to converge.

Exercise 7:

Next we study the behavior of Newton's method for the function $f(x) = x^{1/3}$. This problem can be done algebraically (and in fact, doesn't work numerically for some versions of MATLAB) so we ask it to be done by hand.

a. Write out the equation $x_{n+1} = x_n - f(x_n)/f'(x_n)$ and simplify the right hand side **using algebra**. Write your answer in MATLAB notation. (Don't be surprised if this problem simplifies a lot)
    (20) Answer:

b. Starting with $x_0 = 1$ find the values of x_3 and x_4.
    First, $x_3 = ?$
    (21) Answer: _____

    And $x_4 = ?$
    (22) Answer: _____

3   function file for Newton's method

If you were going to use Newton's method frequently, you would want to write a program to do so. Here is a MATLAB program that performs Newton's method. It assumes you have two function files f.m and fp.m containing functions for $f(x)$ and $f'(x)$.

```
function y = nm(x)
 % nm(x): newton's method. starts at x. y is c
 % assumes f, fp are defined.
  TOL = sqrt(eps);
  MAX_STEPS = 100;
  counter = 0;
  record = [x]; % keep track for the plot
  while ( (abs( f(x))  > TOL ) & (counter < MAX_STEPS))
    x = x - f(x)/fp(x);
    record = [record,x];
    counter = counter + 1;
  end

  y = x;
  % now make a plot
  t =  linspace(min(record)-1,max(record)+1);
  plot(t,f(t),'r'); hold on;
  plot([record(1),record(1)],[0,f(record(1))]); % the first line
  linesx = []; linesy = [];
  for i = 2:counter
    linesx = [linesx,record(i-1),record(i),record(i)];
    linesy = [linesy,f(record(i-1)),0,f(record(i))];
  end;
  plot(linesx,linesy)

  % give the output
  if counter == MAX_STEPS
    fprintf(1, 'Newtons method did not converge in %3.0f steps\n',counter);
  else
    fprintf(1, 'Newtons method converged in %3.0f steps.\n',counter);
    fprintf(1,'The approximate root is %12.8f.\n', x);
  end
```

MTH229

The College of Staten Island
Department of Mathematics

Optimization

1 Introduction

In this project, we use MATLAB to help us with optimization problems. For these problems, we will be finding the global minimum or maximum of a function. Using the zooming technique may work very poorly for this type of exploration, as it is hard to zoom in on the exact point where a function is flat. If this is the case, you can always plot the function's derivative, or approximate derivative, and then zoom in on the critical point associated with the minimum or maximum.

2 Exercises

Exercise 1:
Dimensions of the Largest Box

An open box is to be made from a rectangular piece of cardboard measuring $18" \times 48"$. The box is made by cutting equal squares from each of its 4 corners and turning up the sides. (Suggestion: you can try making one yourself with a spare piece of paper.)
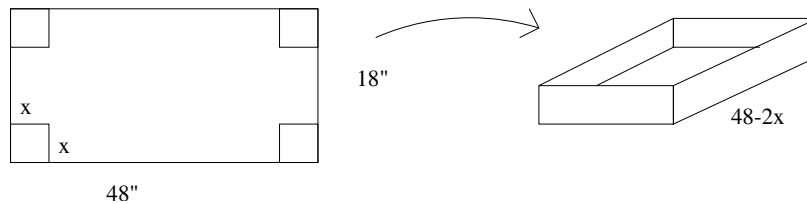
a. Let x be the side of a square removed from each corner. Express the volume V of the box as a function of x. (The length is $48 - 2x$, what is the width?)
$V(x) =$?
(1) Answer:

---

b. What is the domain of possible values for x? (That is, the domain of $V(x)$?)
   (2) Answer:

c. Make a graph of $V(x)$ over the domain chosen. Graphically determine all possible values of x so that the volume of the box is $V = 1400$ cubic inches. Determine these values to an accuracy of 3 significant digits.
   (3) Answer: _____

d. The equation $V(x) = 1400$ is a cubic equation with 3 real roots. Find all three roots using MATLAB's "roots" command. Do you get the same answers as your graphical investigation. Explain.
   (4)

e. Graph $V(x)$ and $V'(x)$ together. Use the graph of $V'(x)$ to find the value of x that maximizes V.
   (5) Answer: _____

f. Find $V'(x)$ analytically by differentiating your formula for $V(x)$. Does your formula show that your last answer is a critical point for $V(x)$?
   (6) Circle one:
   1. yes **2.** no

g. Find the exact value of the maximum of $V(x)$.
   (7) Answer: _____

Exercise 2:
Largest rectangle inscribed in a semicircle
    Determine the area of the largest rectangle that can be inscribed in a semicircle of radius 8". Figure 1 shows that the area can be written as $A = (2x)y$, if (x, y) is the point of the upper right corner of the rectangle. However, we choose to parameterize the area by a single value, the angle θ.
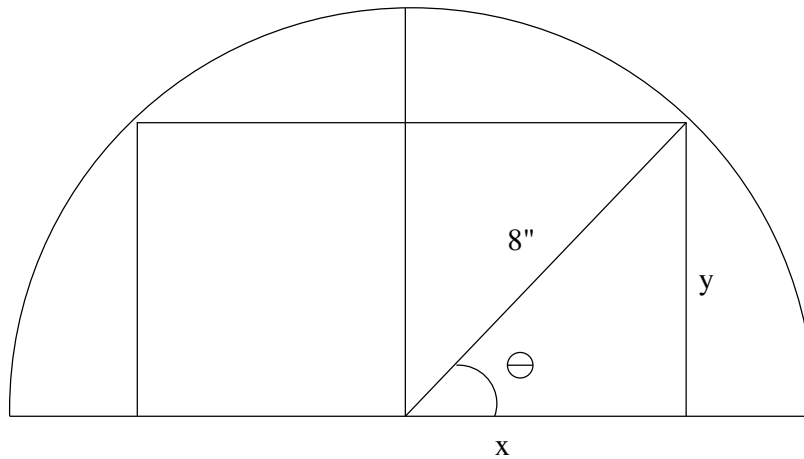

Figure 1: What is the rectangle with largest area that can be inscribed in a half circle?

a. Derive the formula for the area of the inscribed rectangle as a function of θ. We refer to this function as $A(\theta)$ below.

    $A(\theta) =$?

    (8)

b. Plot $A(\theta)$ over its relevant domain. What is the relevant domain?

    (9) Circle one:
    1. $[0, \pi]$
    2. $[0, \pi/2]$
    3. $[0, 8]$
    4. $[0, 4]$
    5. None of these answers

c. For what value of θ is the maximum area attained?
    (10) Answer: _____

d. What is the maximum area?
   (11) Answer: _____

Exercise 3:
A walk in the park?
Eva wants to get to the bus stop as quickly as possible. The bus stop is across a grassy park, 2000
feet west and 600 feet north of her starting position. Eva can walk west along the edge of the park
on the sidewalk at a speed of 6 ft/sec. She can also travel through the grass in the park, but only at
a rate of 5 ft/sec (the park is a favorite place to walk dogs, so she must move with care). What path
will get her to the bus stop the fastest? Remember,

$$\text{distance} = \text{rate} \times \text{time.}$$
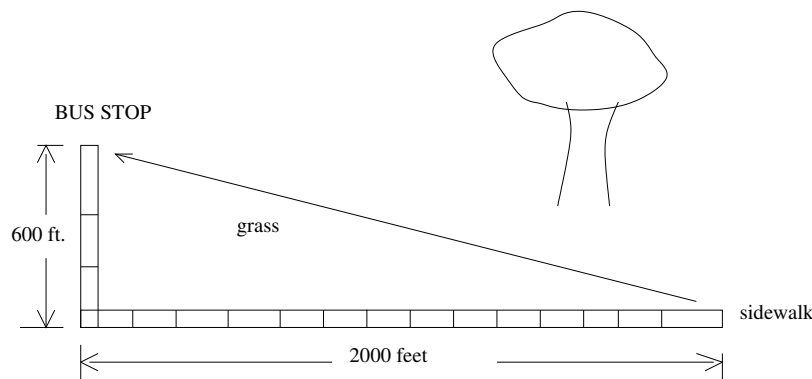
Figure 2: What is fastest way to walk to the bus stop?

a. How long would it take Eva to walk to the bus stop if she doesn't want to get her new shoes
   dirty? (If she only walked on the sidewalk.)
   (12) Circle one:
   1. $t = 2600/6$
   2. $t = 2600/5$
   3. $t = \sqrt{2000^2 + 600^2}/6$
   4. $t = \sqrt{2000^2 + 600^2}/5$
   5. None of these answers

b. How long would it take Eva to walk to the bus stop if she took the route of shortest distance?
(A diagonal through the park.)
(13) Circle one:
1. $t = 2600/6$
2. $t = 2600/5$
3. $t = \sqrt{2000^2 + 600^2}/6$
4. $t = \sqrt{2000^2 + 600^2}/5$
5. none of these answers

c. If Eva walks x feet west along the sidewalk and then walks diagonally through the park, which function below represents the time it will take her to follow this path?
(14) Circle one:
1. $t = x/6 + \sqrt{2000^2 + 600^2}/5$
2. $t = x/5 + \sqrt{2000^2 + 600^2}/6$
3. $t = x/6 + \sqrt{(2000 - x)^2 + 600^2}/5$
4. $t = x/6 + \sqrt{2000^2 + (600 - x)^2}/5$
5. none of these answers

d. Use your answer in part 3 above and MATLAB to determine the shortest time it could take Eva to get to the bus stop. The minimum time is:
(15) Circle one:
1. 360
2. 400
3. 433
4. 417
5. none of these answers

e. How far should Eva walk west along the sidewalk in order to minimize the amount of time it takes her to get to the bus stop?
(16) Circle one:
1. 0
2. 400
3. 1,100
4. 2,000
5. none of these answers

Exercise 4:

The ladder problem

A two-dimensional contractor would like to take a ladder down a hallway, but must negotiate a corner. The dimensions of the hallway are illustrated in Figure 3. That is, one width is 8 feet, the other 5 feet. What is the longest ladder that the contractor can successfully get around the corner and through the hallway?
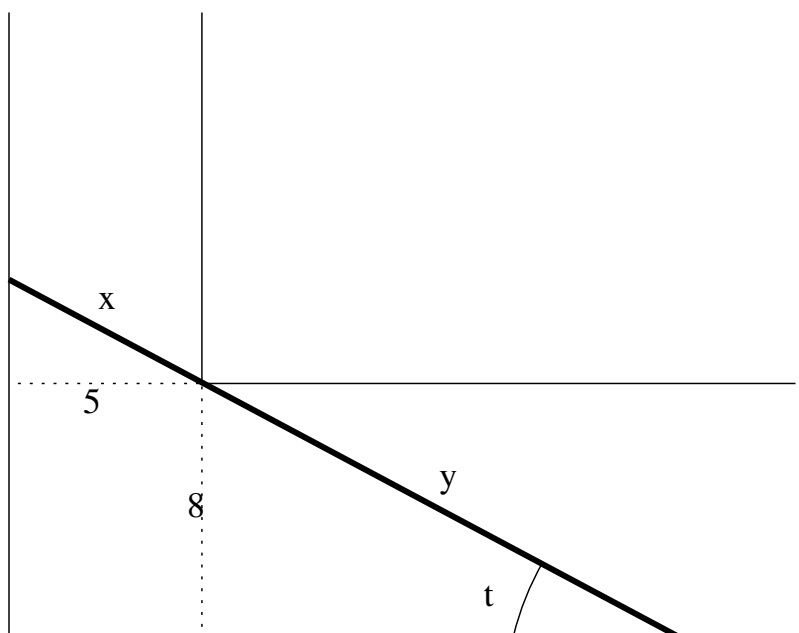

Figure 3: Illustration of the longest ladder, $l(t) = x + y$, that can fit at an angle t.

Figure 3 also has drawn on it the longest possible ladder that can fit for a given angle t. This ladder will touch in three points: the corner, and the two outside edges of the hallway.

We focus on the function $l(t)$ which gives the length of the longest ladder that can fit for a given angle t.

   a. Find a formula for $l(t) = x + y$:

      (17) Circle one:

      1. $l(t) = 5\tan(t) + 8\cot(t)$

      2. $l(t) = 5/\cos(t) + 8/\sin(t)$

      3. $l(t) = 5\cos(t) + 8\sin(t)$

      4. $l(t) = (5^2 + \cos(t))^{1/2} + (8^2 + \sin(t))^{1/2}$

   b. What is the length of the longest ladder that can fit when the angle is $\pi/6$ radians (30 degrees)?

      (18) Answer: _____

c. A 20 foot ladder will get stuck on its way around the corner of the hallway – at one angle if carried through the 8' hallway, and at another angle through the 5' hallway. What are these two angles?

(19) Answer: _____

d. Plot a graph of $l(t)$ over a reasonable viewing window. Find its minimum value and locate the t value for which this happens. What is the angle t? (Answer in radians)

(20) Answer: _____

e. If a ladder is longer than this minimum value, there will be angles for which it won't fit around the corner. For ladders shorter than this minimum value this won't be the case. Use this to find the length of the longest ladder the contractor can carry around the corner.

The longest ladder is:
(21) Answer: _____

MTH229

The College of Staten Island
Department of Mathematics

Definite Integrals and Riemann Sums

1   Introduction

The evaluation of a definite integral is made easy by the fundamental theorem of calculus which states if $F(x)$ is an anti-derivative for a continuous function $f(x)$ then

$$\int_a^b f(x)dx = F(b) - F(a).$$

Of course, this is easy only if you can actually find a function $F(x)$. When you cannot, this theorem provides no help. In this case, when the function $f(x)$ is non-negative we can use the interpretation of the definite integral involving $f(x)$ in terms of the area under $f(x)$ above the x axis between a and b. This was the basis for the Riemann sum definition of the definite integral.

For continuous functions $f(x)$, the theory of Riemann sums implies that

$$\int_a^b f(x)dx = \lim_{n \to \infty} \sum_{i=1}^n f(x_i^*)\Delta x,$$

where the interval $[a, b]$ has been partitioned into n equal-sized subintervals of length $\Delta x = (b - a)/n$ and x_i^* is some value within the ith subinterval. Notationally, if we fix n we can define $a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$ where $x_i = a + i\Delta x$. The value x_i^* may be any value in the ith subinterval $[x_{i-1}, x_i]$. Choosing x_i^* to be x_{i-1} uses the left-hand endpoints. Choosing x_i^* to be x_i uses the right hand endpoints, a choice we will make in the following.

To use MATLAB to investigate limits as n goes to ∞, one may take larger and larger values of n, compute the sum, and then see if the sums appear to converge. If so, we have good reason to believe we have an answer to our definite integral.

1.1   New MATLAB commands

   a. `sum(x)` – add up all the numbers in the list x.

2   Using Riemann sums to approximate the integral

Figure 1 shows a Riemann sum approximation with $n = 5$ for $f(x) = \sin(x)$ over the interval $[0, \pi/2]$. As we can see, there is a lot of extra area. However, as n gets larger, the convergence of the Riemann sum to the definite integral implies this excess area gets smaller and smaller, vanishing in the limit.
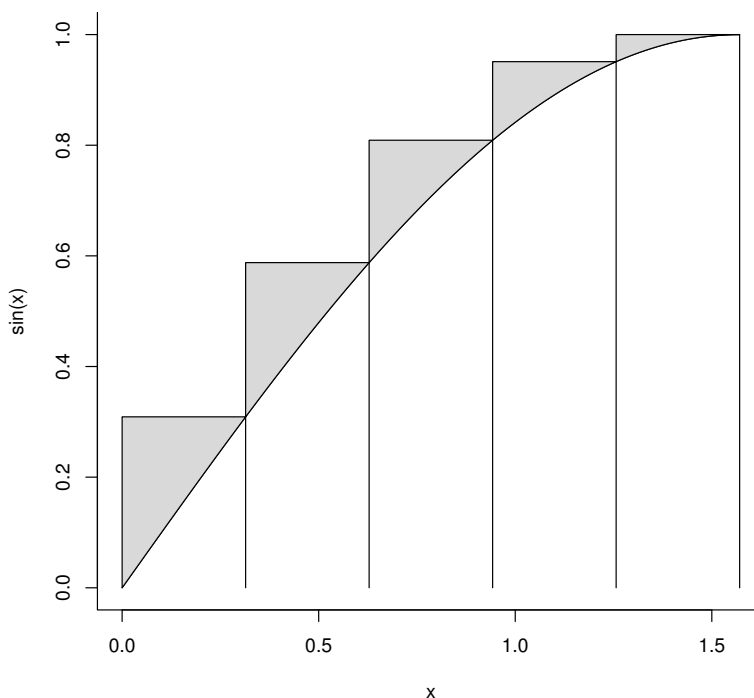
Figure 1: Plot of $f(x) = \sin(x)$ over the interval $[0, \pi/2]$. Layered on top are 5 rectangles whose area yields the Riemann sum for $n = 5$ using right hand end points. Excess area is shaded.

To compute the area using MATLAB can be done by mimicking the formulas above:

```
>> a=0; b = pi/2; n = 5;
>> delta = (b-a)/n;
>> i = 1:n;
>> x = a + i*delta;              % stores x1, x2, ..., xn
>> sum(sin(x) * delta)
ans = 1.1488
```

We don't expect this answer to be close to the correct answer (which we know is $1 = -\cos(\pi/2) + \cos(0)$) as n is quite small. We should explore the sum for different values of n, to see how large n should be to have an accurate approximation. This may be done by taking a larger value of n above, and rerunning the commands. We will show how to do this using a script file.

2.1   Script files

A function file stores the contents of a function definition in a file for later use. A script file stores a sequence of commands for later use. We open a new script file using the menu item `New ->   Script` under the `File` menu of the main menubar. This is just like when we open a function file.

Instead of starting with the keyword `function`, we simply type the commands we wish to have executed. In this case, with $n = 250$, we type in (without prompts):

```
a=0; b = pi/2; n = 250;
delta = (b-a)/n;
i = 1:n;
x = a + i*delta;               % stores x1, x2, ..., xn
sum(sin(x) * delta)
```

We save this with the file name `int.m` in the default directory. Then whenever we issue the command `int`, the commands in this file will be executed. So to find the value when $n = 250$ is now as easy as entering:

```
>> int
ans = 1.0031
```

This is pretty close to the answer, but we would like 4 decimal points of accuracy, not just 2. How large must n be to achieve that?

Exercise 1:

a. Create a script file containing the content above, only change the value of n to 2,500. What is the area for the Riemann sum now?
   (1) Answer: _____

b. The advantage of script files is that they can be easily edited and run. Simply change what you want, save your work, and call it from the command line. Try all of these values of n: 7,500, 15,000 and 30,000 until you find a value where the Riemann sum is 1 to 4 decimal points. What value of n do you find?
   (2) Circle one: 1. 7,500 **2.** 15,000 **3.** 30,000

You may also wish to use a function file instead of a script file. As well, using a function file to evaluate the function can be convenient.

Exercise 2:
We will now compare the area under $\sin(x)$ to that under $f(x) = \sin(\sqrt{x})$. Figure 2 shows graphs of both functions over $[0, \pi]$.

```
>> x = linspace(0,pi); plot(x,sin(x),x,sin(sqrt(x)));grid
```
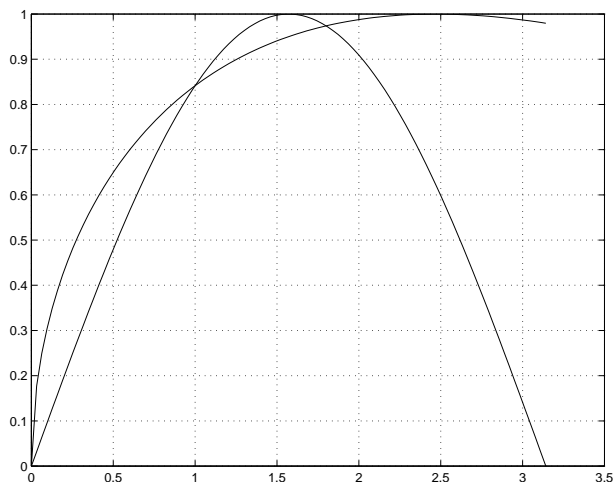

Figure 2: Graphic comparing area under $\sin(x)$ and $\sin(\sqrt{x})$ between $[0, \pi]$.

We see that the area under the $\sin(\sqrt{x})$ curve looks larger than 2 — the area under $\sin(x)$. We find the actual area to be certain. We don't know an anti-derivative to this function so the fundamental theorem of calculus will be of no help. Rather, use the Riemann sum approximation to find the area associated to the definite integral

$$\int_0^\pi \sin(\sqrt{x})dx.$$

a. Find the value of $\sum_{i=1}^n f(x_i)\Delta x$ for $n = 50$.
   (3) Answer: _____

b. Find the value of $\sum_{i=1}^n f(x_i)\Delta x$ for $n = 500$.
   (4) Answer: _____

Somewhere between $n = 25,000$ and $n = 30,000$ the value stabilizes to 4 digits, the value being 2.6695. Again, we see that this can be a slow process to converge.

Now that you know how big n generally needs to be for a fairly smooth function to get 4 digits of accuracy using Riemann sums, we will just set $n = 30,000$ in the following.

Exercise 3:

The function $e^{-x^2/2}$ is very important in probability theory. Figure 3 shows the graph. To help estimate the area, two lines forming a triangle with the x axis are added
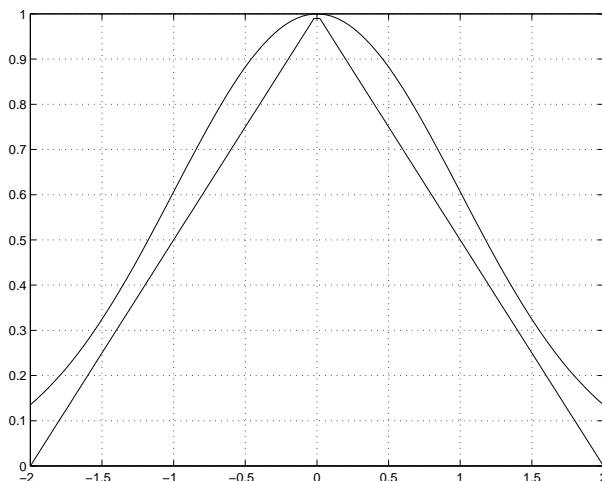

Figure 3: Graph of bell-shaped curve. Straight lines are added to estimate the area under the bell.

The triangle has area 2. From the graph then we know our answer should be slightly bigger than 2. Let's see what it is.

a. From the graph explain why

$$\int_{-2}^{2} e^{-x^2/2}dx = 2 \int_{0}^{2} e^{-x^2/2}dx$$

(5)

b. Use $n = 30,000$ to estimate using a Riemann sum

$$\int_{-2}^{2} e^{-x^2/2}dx.$$

(Here we have a different from 0.) **(6) Answer:** _____

Exercise 4:

Find the approximate value of the integral

$$\int_0^\pi \frac{\sin x}{x} dx$$

by using a Riemann sum approximation with $n = 30,000$. (We avoid issues with the definition of the Riemann sum by assuming $f(0) = 1$, so that $f(x)$ is continuous.)

(7) Answer: _____

Exercise 5:

Many everyday vessels containing fluid are mathematically known as surfaces of revolution, as their surface can be created by rotating the graph of some function $f(x)$ around the x axis. For instance, the linear function $f(x) = 2.5 + x/10$, $0 \le x \le 15$ when rotated around the x axis, traces out a glass shape that is roughly the size of a 16-ounce tumbler. This is illustrated in Figure 4 with the x-axis running vertically instead of the more traditional horizontal direction.
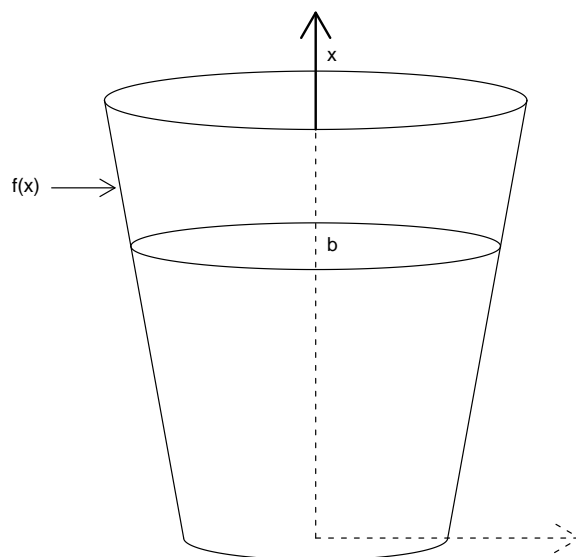

Figure 4: A tumbler described by $f(x)$ filled to a height of b.

The exact volume of fluid in the vessel depends on the height to which it is filled. If the height is labeled b, then the volume is

$$V(b) = \int_0^b \pi f(x)^2 dx.$$

a. Find the volume contained in the glass if it is filled to the top $b = 14$ cm. This will be in metric units of cm^3. To find ounces divide by 1000 and multiply by 33.82.

How many ounces does this glass hold?
(8) Answer: _____

b. Just what is meant by "the glass is half full?" If the glass is filled to $b = 7$ cm, what percent of the total volume is this?

Answer with a percent (Volume for 7/Volume for 14 times 100).
(9) Answer: _____

c. Now, by trying different values for b, find a value of b within 1 decimal point (eg. 7.4 or 9.3) so that filling the glass to this level gives half the volume of when it is full.

$b = ?$
(10) Answer: _____

Exercise 6:
The area under a curve $f(x)$ is given by the definite integral. But what about the length of the curve? This too is answered by a definite integral, but with a different formula:

$$\text{Length} = \int_a^b \sqrt{1 + (f'(x))^2} dx :$$

Let's use this formula to compute the length of the three curves shown in Figure 5.

a. Let $f(x) = x^2$. Find the length of the graph of $f(x)$ over the interval $[0, 2]$. (It should be between $4.472 = \sqrt{2^2 + 4^2}$ and $6 = 2 + 4$.)
(11) Answer: _____

b. Let $f(x) = e^x$. Find the length of the graph of $f(x)$ between $x = 0$ and $x = 2$. (It should be between $6.695 = \sqrt{2^2 + (e^2 - 1)^2}$ and $8.389 = 2 + (e^2 - 1)$.
(12) Answer: _____

c. Let $f(x) = \ln(x)$. (This has derivative $1/x$.) Find the length of this curve from $1 = e^0$ to e^2. (The answer might be expected, given your last, after a bit of thought.)
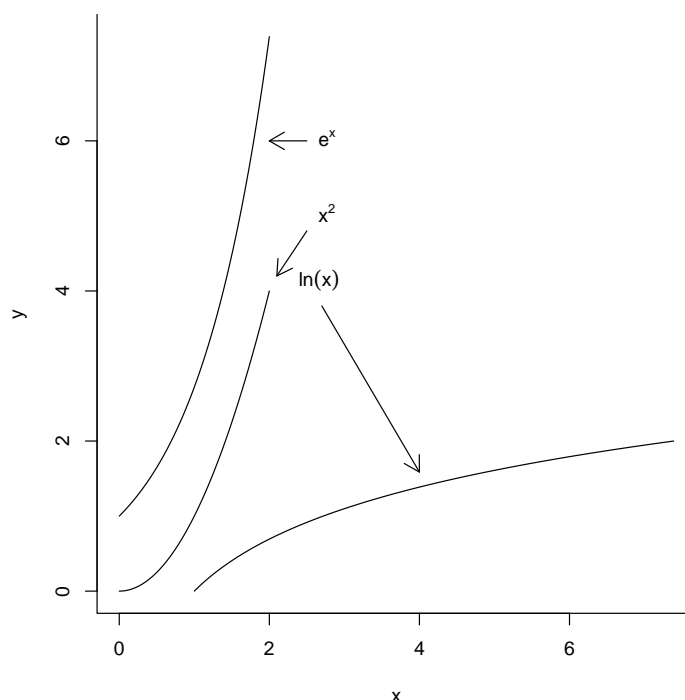**(13) Answer:** _____

Figure 5: What is the length of the three curves?

3   The trapezoid method

The convergence of the Riemann sum is quite slow. A way to speed this up is with the trapezoid method. The trapezoid method replaces the rectangles in the sum above with trapezoids. They are formed with the two endpoints. Figure 6 illustrates the savings in excess area.

The area of the trapezoid is the average height times the base, or

$$\frac{f(x_i) + f(x_{i-1})}{2}\Delta x.$$

Adding up the areas of these trapezoids gives an approximation to the definite integral. Here are MATLAB commands that do this for the $\sin x$ example

```
>> a=0;  b=pi/2; n = 5;
>> delta = (b-a)/n;
>> xi = a + (1:n)*delta;              % x_i above
>> ximinus1 = a + (0:(n-1))*delta;    % x_(i-1) above
>> y = (sin(xi) + sin(ximinus1))/2;   % the average height of the trapezoid
>> sum(y*delta)                       % the sum of the area of the trapezoid
   ans = 0.9918
```
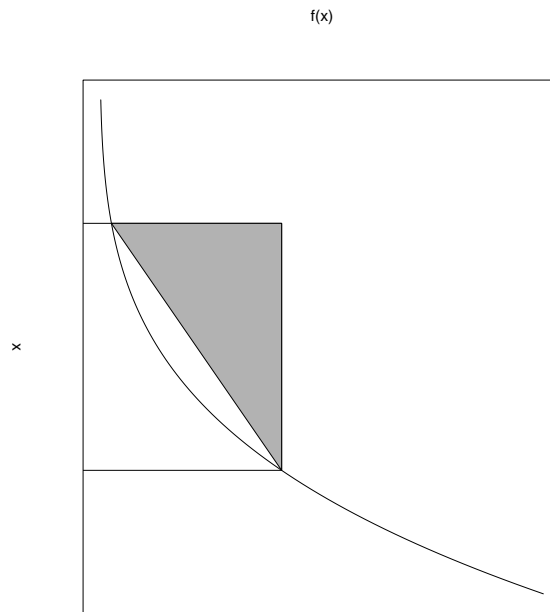
f(x)

x

Figure 6: Illustration comparing trapezoid method to Riemann sum using rectangles.

This is much closer to the known value of 1.0 than that of 1.1488 given by the Riemann sum.

Exercise 7:

Do the trapezoid method above for $n = 50$ to find an approximate answer for the definite integral

$$\int_0^\pi \sin(\sqrt{x})dx$$

a. **(14) Answer:** _____

b. Has the method given the answer 2.6695 by the time $n = 1000$? (Remember it took an n between 25,000 and 30,000 using rectangles.)
   (15) Circle one: 1. Yes **2.** No

    For reference here is a function file that you can use to evaluate an integral using the trapezoid method. To use it you need first to make a function file called f.m that contains your function.

```
function area = integrate(a,b,n)
%    integrate(a,b,[n])
%    integrates the function called 'f' in the file f.m over
%    the interval [a,b] with n subintervals using the trapezoid
%    method. The default for n is 100.

  if nargin < 2
    error('Called improperly. Try help integrate');
  elseif nargin < 3
    n = 100;              % default number
  end


  delta = (b-a)/n;
  x1 = a + (1:n)*delta;
  x0 = a + (0:(n-1))*delta;
  area = sum( (f(x1) + f(x0))/2 ) * delta;
```