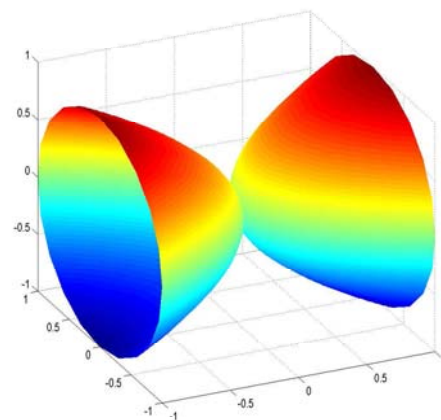


The College of Staten Island
Department of Mathematics



MTH 232

Calculus II

<http://www.math.csi.cuny.edu/matlab/>

MATLAB PROJECTS

STUDENT: _____

SECTION: _____

INSTRUCTOR: _____

BASIC FUNCTIONS

Elementary Mathematical functions		
MATLAB notation	Mathematical notation	Meaning of the operation
<code>sqrt(x)</code>	\sqrt{x}	square root
<code>abs(x)</code>	$ x $	absolute value
<code>sign(x)</code>		sign of x (+1, -1, or 0)
<code>exp(x)</code>	e^x	exponential function
<code>log(x)</code>	$\ln x$	natural logarithm
<code>log10(x)</code>	$\log_{10} x$	logarithm base 10
<code>sin(x)</code>	$\sin x$	sine
<code>cos(x)</code>	$\cos x$	cosine
<code>tan(x)</code>	$\tan x$	tangent
<code>asin(x)</code>	$\sin^{-1} x$	inverse sine
<code>acos(x)</code>	$\cos^{-1} x$	inverse cosine
<code>atan(x)</code>	$\tan^{-1} x$	inverse tangent

MTH232

The College of Staten Island
Department of Mathematics

Introduction to the Symbolic Math Toolbox

New Symbolic MATLAB commands:

```
syms
solve(f)
subs(f,x,'b')
simplify(f)
diff(f,x)
ezplot(f)
```

In this project, we introduce *Symbolic Math* using MATLAB’s *Symbolic Math Toolbox* along with *Maple Symbolic Functions*. We begin our explanation by way of review of elementary MATLAB syntax.

1 Symbolic vs. Numeric Operations in MATLAB

When you issue the command

```
>> a = 2
```

the letter “a” becomes a *numeric variable* to which the value 2.0 is assigned. As we learned in the Calculus I lab, MATLAB operates in this way as a simple calculator. If you type

```
>> a+a
```

MATLAB’s response is

```
ans = 4
```

This is a *numeric operation* performed on a *numeric variable*. But suppose now that we want to perform a “*Symbolic Operation*”, that is, we would like to type “b+b” and have the computer respond with “2*b”, without first assigning values to the variable b.

```
>> b+b
```

```
??? Undefined function or variable 'b'
```

If we want MATLAB’s response to be “2*b”, we have to first define b as a *Symbolic Variable*. How can MATLAB perform operations symbolically?

1.1 MATLAB’s Symbolic Math Toolbox

As it turns out, MATLAB can perform *Symbolic Arithmetic* through the use of its **Symbolic Math Toolbox** – an add-on to MATLAB – which is based upon a software program called **Maple**, a product of Waterloo Maple Software, Inc. Let’s see how symbolic variables are defined in MATLAB.

◇ **The syms Command** – defines a symbolic variable

To get MATLAB to add b+b symbolically, we need first to define b as a **symbol**.

```
>> syms b
```

Now we can perform b+b symbolically.

```
>> b+b
ans = 2*b
```

(To display all currently defined symbolic variables, just type the word `syms`, or type the word `who` to display all numeric as well as symbolic variables.)

There are many instances in which symbolic tools are useful.

1.1.1 Expression Simplification and Minimization of Round-off Error

Irrational Numbers and Numerical Values

Long ago, it was proved that the square root of any prime number is irrational, that is, non-terminating. For instance, $\sqrt{7} = 2.6458\dots$. The dots specify that an infinite chain of digits exists to the right of the decimal point having *no* pattern or repeating sequence! Hence, when performing calculations on expressions containing irrational numbers, the result is always rounded – an *error* always exists. For example, when typing in `sqrt(7)`, MATLAB responds with a rounded estimate, 2.6458. Furthermore, needlessly complex expressions will introduce excessive error. For instance, try evaluating $7/\sqrt{7}$ (which we know is equal to $\sqrt{7}$) numerically.....

```
>> format long
>> 7/sqrt(7)
ans = 2.645751311064590    % this is less accurate
>> sqrt(7)
ans = 2.645751311064591    % this is more accurate
>> format short           % go back to four place accuracy
```

Evidently, the division operation produced a small error, on the order of 10^{-13} . Thus, we can see that it is advantageous (if we are interested in extreme precision!) to simplify expressions before doing calculations. When employing symbolic operations, however, the expressions are often simplified – avoiding the round-off error that would otherwise result. Let’s define the above expression symbolically.

```
>> syms x
>> 7/sqrt(7)*x
ans = 7^(1/2)*x
```

(Symbolic expressions must include symbolic variables, and not just constants.)

It can be seen in this case that $7/\sqrt{7}$ as part of a symbolic expression produced immediate simplification: $7/\sqrt{7}$ was replaced by $\sqrt{7}$. However, not all symbolic expressions are automatically simplified.

◇ The `simplify()` and `simple()` Commands – Simplify symbolic expressions

Example 1:

Consider the trigonometric identity $\sin^2 x + \cos^2 x = 1$.

It is important to note at this point that we never use the DOT “.” operator in symbolic math.

```
>> syms x
>> f = sin(x)^2 + cos(x)^2 % note the absence of the DOT operator!
```

MATLAB responds with

```
f =
cos(x)^2 + sin(x)^2
```

First, we defined x to be a symbolic variable. Next, we defined f to be a symbolic function of x . In this case, however, $f(x)$ does not automatically simplify to 1. If we want to simplify this function...

```
>> simplify(f)
ans =
1
```

Thus, the Symbolic Toolbox has confirmed that $\sin^2 x + \cos^2 x = 1$.

(In cases where `simplify` doesn’t seem helpful, try `simple` instead.)

Finding Zeros of a Function Symbolically

Recall that in the Calculus I Computer Lab you learned how to find zeros using the `roots` command, or, perhaps you graphed the function and zoomed in on the zeros. You also may have learned to use Newton’s Method to accomplish the task. The values you arrived at were decimal numbers which sometimes were estimates of irrational roots.

◇ **The solve() Command** – Symbolically solves $f(x) = 0$

◇ **The subs(f,x,'b') Command** – symbolic function evaluation

- where f is defined as a symbolic function of x , and we want to evaluate $f(b)$
- enclose b in single quotes if it is a constant.

Example 2:

Find the zeros of $f(x) = x^2 + 4x + 2$

First, recall the calculus 1 Non-Symbolic method:

```
>> p=[1 4 2]
>> roots(p)
ans =
    -3.4142
    -0.5858
```

When solving $f(x) = x^2 + 4x + 2 = 0$ by hand using the quadratic formula, you find the roots in the form of $-2 \pm \sqrt{2}$, and *not* in the inexact decimal form “-3.4142 and -0.5858”. The Symbolic Toolbox produces the same solutions as those worked out by quadratic formula, and is performed by the following commands....

```
>> syms x % defines x to be a symbolic variable.
>> f=x^2+4*x+2 % Assigns the named function to symbol f. No dots!
>> solve(f) % sets f(x) = 0 and solves for x
ans=
    - 2^(1/2) - 2
    2^(1/2) - 2
```

Thus $-2 \pm \sqrt{2}$ are the zeros of $f(x)$. This is more exact information than -3.4142 and -0.5858. We can check our work by evaluating $f(-2 + \sqrt{2})$ and $f(-2 - \sqrt{2})$ symbolically.

```
>> subs(f,x,'-2-2^(1/2)') % single quotes around -2 - sqrt(2)
ans =
    (2^(1/2) + 2)^2 - 4*2^(1/2) - 6
>> simplify(ans)
ans =
    0
>> simplify( subs(f,x,'-2+2^(1/2)') ) % f(-2 + sqrt(2))
ans=
    0
```

Explanation: solve(f) solves $f(x) = 0$ for x . As you have seen, numbers are left in symbolic form. Radicals are left alone – no decimal numbers are produced, and no rounding occurs. The command subs along with simplify confirms that $f(-2 - \sqrt{2}) = f(-2 + \sqrt{2}) = 0$.

Exercises

Exercise 1:

Find the zeros of $f(x) = x^3 - x^2 - 18$ symbolically.

- First, what MATLAB command defines x to be a symbolic variable?

(1) Answer:

- What command defines $f(x)$ symbolically?

(2) Answer:

- What command symbolically determines the zeros of $f(x)$?

(3) Answer:

- How many real roots are there?

(4) Circle one:

1. 1
2. 2
3. 3
4. 4

- What are the zero(s) of $f(x)$?

(5) Circle one:

1. 3 and $-1 \pm \sqrt{5}$
2. $-1 \pm \sqrt{5}$
3. 3
4. $2 \pm \sqrt{2}$

- What command verifies your results, that is, evaluates $f(a)$ where a is one of the roots?
(6) Answer:

Exercise 2:

The *linear approximation*, $P_1(x)$, of a function $f(x)$ is defined as

$$P_1(x) = f(a) + f'(a)(x - a)$$

Further, the *quadratic approximation* is defined as

$$P_2(x) = f(a) + f'(a)(x - a) + 1/2 f''(a)(x - a)^2$$

Use symbolic math to find $P_1(x)$ and $P_2(x)$ if $f(x) = \arcsin x$ and $a = 1/2$ (note: $\arcsin x$ in MATLAB is `asin(x)`)

For the following: use fractions, not decimal notation, for constants.

- Assume `syms x` has been entered.
- What MATLAB command defines $f(x) = \arcsin x$ symbolically?
(7) Answer:
- What MATLAB command finds $f'(x)$, and assigns it to a variable called `fp`?
(8) Answer:

- What command evaluates $f(1/2)$ symbolically, and assigns it to a variable named f1?
(9) Answer:

- What command evaluates $f'(1/2)$ symbolically, and assigns it to a variable named fp1?

(10) Circle one:

1. `fp1=simplify(fp,x,'1/2')`
2. `fp1=solve(fp,x,'1/2')`
3. `fp1=subs(fp,x,'1/2')`
4. `subs(fp,x,'1/2')`

- With all the above commands typed in ,what command would you now use to define $P_1(x)$ in MATLAB, call it p1?

(11) Circle one:

1. `p1=fp1*(x-1/2)`
2. `p1=f1+fp1(x-1/2)`
3. `p1=f1+fp1*x-1/2`
4. `p1=f1+fp1*(x-1/2)`

- $P_1(x)$, when *simplified*, is equal to...

(12) Circle one:

1. $1/6 \pi + \sqrt{3}(2/3 x - 1/3)$
2. $\pi + \sqrt{2}(2/3 x - 1/3)$
3. $1/6 \pi + \sqrt{2}(2/3 x - 1/3)$
4. $\sqrt{3}(2/3 x - 1/3)$

- What command defines fpp as the second derivative of $f(x)$? (Hint: take the derivative of fp)

(13) Answer:

- What command evaluates $f''(1/2)$ symbolically, and assigns it to a variable named fpp1?

(14) Circle one:

1. `fpp1=simplify(fpp,x,'1/2')`
2. `fpp1=solve(fpp,x,'1/2')`
3. `fpp1=subs(fpp,x,'1/2')`
4. `subs(fpp,x,'1/2')`

- What command would you use to define $P_2(x)$ in MATLAB, call it p2?

(15) Circle one:

1. `p2=f1+fp1*(x-1/2)`
2. `p2=f1+fp1*(x-1/2)+1/2*fpp1*(x-1/2)^2`
3. `p2=f1+fp1*(x-1/2)^2+1/2*fpp1*(x-1/2)`
4. `p2=1/2*fpp1*(x-1/2)^2`

- $P_2(x)$, when simplified, is equal to....

(16) Circle one:

1. $\frac{1}{6}\pi - \frac{5}{18}\sqrt{3} + \frac{2}{9}\sqrt{3}x^2$
2. $\frac{1}{6}\pi + \frac{4}{9}\sqrt{3}x - \frac{5}{18}\sqrt{3} + \frac{2}{9}\sqrt{3}x^2$
3. $\frac{1}{6}\pi + \frac{4}{9}\sqrt{3}x + \frac{2}{9}\sqrt{3}x^2$
4. $\frac{2}{9}\sqrt{3}x^2 - \frac{5}{18}\sqrt{3}$

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Applications of Definite Integration Using the Symbolic Math Toolbox

MATLAB commands:

```
ezplot(f,[a,b])  
text(x,y,'  ')  
gtext('  ')  
diff(f,2)  
int(f,a,b)
```

```
single(  )
```

In this project, we will introduce some basic symbolic commands involving finding zeros, graphing, substitutions, differentiation and integration.

1 Finding Zeros of a Function Symbolically

The command `solve(f)` means solve $f(x) = 0$. It should be noted immediately that **NO** dots need to be used such as have been used previously for multiplication, division and exponentiation.

Example 1:

Find the zeros of $f(x) = x^2 + 2x - 8$

```
>> syms f x      % defines f and x to be “symbols”.  
>> f=x^2+2*x-8  % Assigns the named function to symbol f. Note that there are no dots  
>> solve(f)      % sets f(x) = 0 and solves for x  
ans=  
-4  
 2
```

Thus 2 and -4 are the zeros of $f(x)$.

NOTE:

- “`syms f x`” only needs to be typed once during a MATLAB session. If you type it again after `f=x^2+2*x-8` you lose the value of f .
- If you want, at any time, to know what your defined symbols are, just type `syms` for a symbol listing, or just type `f` or `x` to see its value.

Example 2:

Find the zeros of $g(x) = x^4 - 4x^2$

```
>> syms g x
>> g=x^4 - 4*x^2
>> solve(g)
```

Ans = -2 0 0 2

Exercise 1:

Find all the zeros of $f(x) = 4 * x^3 - x^2 - 4 * x + 1$

(1) Answer: _____

Example 3:

Use MATLAB to find

- a.) all critical numbers
- b.) absolute max and min of $f(x) = 3x^4 - 4x^3 + 5$ on $[-1, 2]$
- c.) the graph of $f(x)$ and label the coordinates of all endpoints and critical numbers

part a:

```
>> f=3*x^4-4*x^3+5 % note that syms f x have already been defined
>> fp=diff(f) % diff symbolically finds f'(x)
>> solve(fp)
ans=
    0
    0
    1
```

Hence, there are 2 critical numbers, $x = 0$ and $x = 1$

part b:

```
>> syms y y1 y2 y3 y4 % do not redefine f or you will reset its value
>> y=subs(f, x) % will substitute the value of x into f(x) and calculate the
corresponding value of y. We will now compute the matching
y value for each endpoint and critical number.

>> y1=subs(f,-1) % → y1 = 12
>> y2=subs(f,0) % → y2 = 5
>> y3=subs(f,1) % → y3 = 4
>> y4=subs(f,2) % → y4 = 21
```

x	-1	0	1	2
y	12	5	4	21

Thus, we have found an absolute max at (2,21) and absolute min at (1,4).

part c:

to graph symbolically, we can use either

- “ezplot(f)”
or
- “ezplot(f, [a,b])” where [a,b] specifies the domain of $f(x)$

try

```
>> ezplot(f)
now try
>> ezplot(f, [-1,2])
```

Observe the difference between them!

For our example, to locate (2,21) it will be useful to adjust the graph using the AXIS ([xmin xmax ymin ymax]) command.

```
>> axis([-1 2 0 25])
```

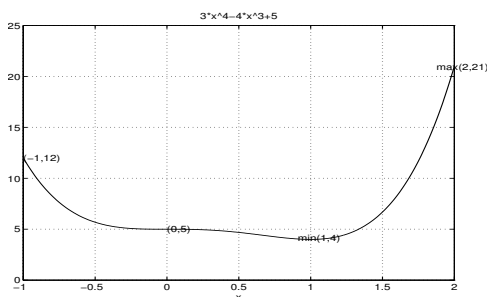
To mark the coordinates (or any text) on a graph use this format:

- “text(a,b, ' ')” places text between single quotes at (a,b) or
- “gtext(' ')” places text *between single quotes* at any location where you click the mouse

When you use gtext and press enter, a cross-hair is shown that can be positioned with the mouse. When a particular location has been selected, press the mouse button and the text will get printed on the graph at the selected location.

Now, let us mark the above mentioned 4 points on the graph.

```
>> text(-1,12, '(-1,12)') % prints (-1,12) on the graph at (-1,12). Note the single quotes.
>> gtext('max(2,21)') % prints max(2,21) at where you position the mouse
```



Use text to label (0,5) and gtext to label min(1,4).

Exercise 2:

Use MATLAB to find the critical numbers, absolute extrema, graph with key points labeled for $f(x) = x^2e^{x/3}$ on interval $[-8,2]$. Submit the completed graph.

a.) The critical numbers are:

(2) Answer: _____

b.) *Absolute* max:

(3) Circle one:

1. the point $(0, 0)$
2. the point $(-6, 36e^{-2})$
3. the point $(2, 4e^{2/3})$
4. none of the above

c.) Absolute min:

(4) Circle one:

1. the point $(0, 0)$
2. the point $(-6, 36e^{-2})$
3. the point $(2, 4e^{2/3})$
4. none of the above

d.) Submit the graph.

(5) Attach your graph to the worksheet.

Exercise 3:

Given, $f(x) = x^4 + 4x^3 + 4x^2 + 4$.

Use MATLAB to find:

a.) zeros

(6) Circle one:

1. where $x=0$
2. where $x=1$
3. there are no zeros
4. none of the above

b.) critical numbers

(7) Answer: _____

c.) Relative max using the Second Derivative Test. (Note that fpp can be found by using either $\text{diff}(fp)$ or $\text{diff}(f,2)$. The 2 in $\text{diff}(f,2)$ means differentiate twice).

$x=$

(8) Answer: _____

d.) Relative min(s)? $x =$

(9) Answer: _____

e.) Find $f''(x_{r.max}) =$

(10) Answer: _____

f.) Find $f''(x_{r.min}) =$

(11) Answer: _____

g.) Submit the graph labeled with min and max. (use “text” command.)

(12) Attach your graph to the worksheet.

2 Finding Points of Intersection and Areas

Given 2 functions $f(x)$ and $g(x)$, at any point of intersection $f(x) = g(x)$ so that $f(x) - g(x) = 0$. Accordingly, the zeros of $y = f(x) - g(x)$ will give us the points of intersection.

Example 4:

Given $f(x) = 2 - x^2$ and $g(x) = x$ Find:

(a) graph of $f(x)$ and $g(x)$

(b) points of intersection and label them on the graph

(c) the area between $f(x)$ and $g(x)$

part a:

```
>> syms f g x % define f,g and x, unless previously defined
>> f=2-x^2
>> g=x
>> ezplot(f)
>> hold on
>> ezplot(g)
>> grid % we must add grid because the second graph eliminates the grid
```

part b: Since $f(x)$ is above and $g(x)$ is below, we will let $y = 2 - x^2 - x$

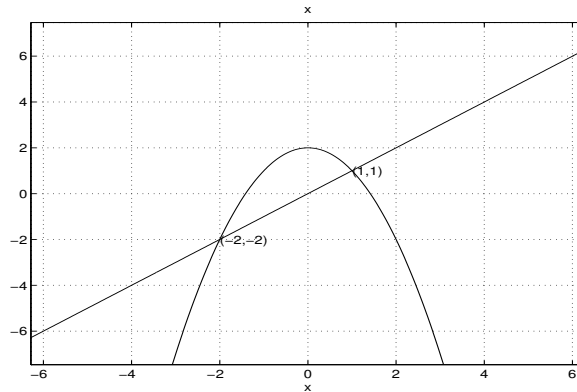
```
>> syms y % Defines symbol y, but does not eliminate f, g or x
>> y = f - g % Note that we can use regular arithmetic operators to
manipulate functions!!
```

```
>> solve(y)
ans = -2 , 1
```

“Applications of Definite Integration Using the Symbolic Math Toolbox”

These are the zeros of y and the x values of the points of intersection. Now let us find the corresponding values of y

```
>> y1=subs(g,-2) % → -2 you could of course use  $f = 2 - x^2$  as well  
>> y2=subs(g,1) % → 1
```



Use text or gtext to label (-2,-2) and (1,1) on the graph

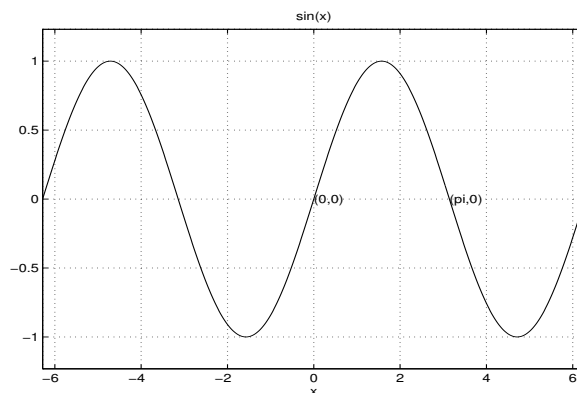
part c: To find the area, we recall that $\text{area} = \int_{-2}^1 (2 - x^2 - x) dx$ (which we defined above, with the statement “ $y=f-g$ ”

```
>> area = int(y,-2,1) % definite integral of y from -2 to 1  
area=9/2
```

Example 5:

Graph and find the area between $y = \sin(x)$ and the x - axis from $x = 0$ to $x = 2\pi$.

```
>> ezplot(sin(x))
```



Since $\text{area} = \int_0^{2\pi} \sin x dx$

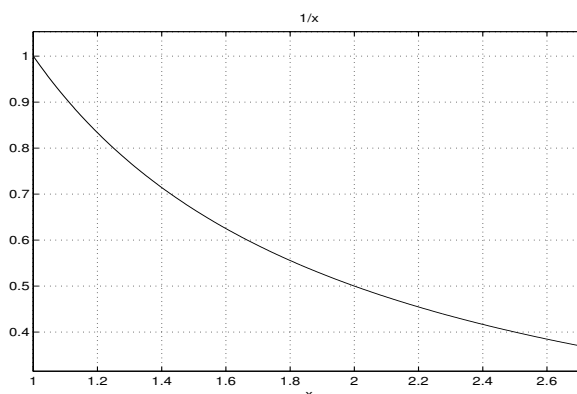
```
>> area = int(sin(x),0,2*pi)
```


area=0

(Notice the area is 0 as you would guess.)

Example 6:

If answers are too clumsy change to numeric answer form. Suppose we are asked to graph and find the area under $f(x) = 1/x$ from $x = 1$ to $x = e$



```
>> syms f x
>> f = 1/x
>> ezplot (f,[1,exp(1)])
>> area= int(f,1,exp(1))
area= log(3060513257434037)-50*log(2)
>> single(area)
ans = 1
```

Exercise 4:

Use MATLAB to graph, find and label points of intersection and determine the area between $f(x) = x^2 + x + 8$ and $g(x) = x + 12$.

a.) Points of intersection (x coordinates):

x=

(13) Answer: _____

b.) the area is:

(14) Answer: _____

c.) Submit the graph

(15) Attach your graph to the worksheet.

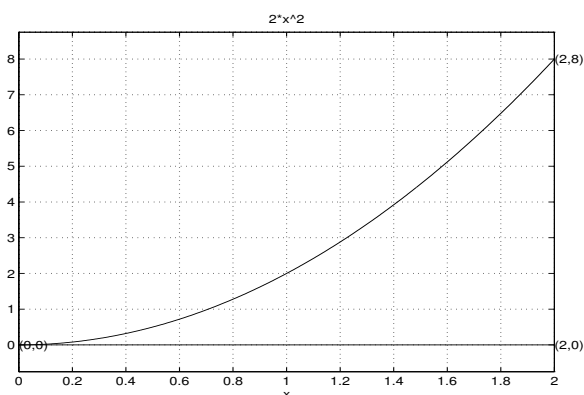
3 VOLUMES AND ARC LENGTH

Example 7:

- Graph the region bounded by $y = 2x^2$, $y = 0$ and $x = 2$.
- Find the volume when the region is rotated around the x-axis.

part a:

```
>> syms x x1
>> x1=sym(0)           % functions which are constants must be defined with sym
>> ezplot(x1)         % graph of y = 0
>> hold on
>> ezplot(2*x^2, [0,2])
>> grid
```



Part b: Disc Method: $\text{Volume} = \pi \int_0^2 (2x^2)^2 dx$

```
>> volume=int(pi*(2*x^2)^2,0,2)
volume = (128*pi)/5
>> single(volume)
ans = 80.4248
```

Shell Method: $\text{Volume} = 2\pi \int_0^8 y(2 - \sqrt{y/2}) dy$

```
>> syms y
>> volume=int(2*pi*y*(2-sqrt(y/2)),0,8)
volume = (128*pi)/5
>> single(volume)
ans = 80.4248
```

Example 8:

Find the volume for the previous problem if the region is rotated around the y-axis.

Disc Method: $\text{Volume} = \pi \int_0^8 2^2 - (\sqrt{y/2})^2 dy = \pi \int_0^8 4 - (y/2) dy$

```
>> volume=int(pi*(4-(y/2)),0,8)
volume = 16*pi
```

Shell Method: $\text{Volume} = 2\pi \int_0^2 x(2x^2) dx$

```
>> volume=int(2*pi*x*2*x^2,0,2)
volume = 16*pi
```

Example 9:

Find the arc length on $f(x) = x^{3/2} - 1$ from $x = 0$ to $x = 4$

```
>> syms f x
>> f = x^(3/2) - 1
>> fp = diff(f) % --> 3/2 * x ^ (1/2)
```

$$\text{arc length} = \int_0^4 \sqrt{1 + (3/2x^{1/2})^2} dx$$

```
>> arcl=int(sqrt(1+(3/2*x^(1/2))^2),0,4)
arcl = (80*10^(1/2))/27 - 8/27
```

For a simpler numeric answer

```
>> single(arcl)
ans = 9.0734
```

Exercise 5:

(a) Graph the region bounded by $y = \sqrt{x+5}$, x -axis, $x = 1$ and $x = 5$

Submit the Graph:

(16) Attach your graph to the worksheet.

(b) Find the volume when the region is rotated around the x-axis.

(17) Circle one:

1. 16π 2. 32π 3. 14π 4. 18π

Exercise 6:

The region bounded by $y = x^3$ and $y = 2x^2$ is rotated around the y-axis.

(a) graph the region. *Note: do not plot the functions outside of the intersection points!*

(18) **Attach your graph to the worksheet.**

(b) find the volume

(19) Circle one:

1. 16π 2. $8\pi/5$ 3. $8\pi/3$ 4. $16\pi/5$

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Integration

1 MATLAB Commands Explained

```
x=sym('x') % same result as using syms x
f=         - no dots!
simple(f)
rsums
```

In this project we study different ways of evaluating integrals. First we see how to find antiderivatives with the help of MATLAB. We then discuss the question of what is to be done when the function has no antiderivative in terms of elementary functions. We'll see how in that case we can use MATLAB to find a numeric approximation to the definite integral.

This project is also designed to help you review methods of integration, both with and without the aid of computers.

2 Finding antiderivatives and definite integrals.

Example 1:

$\int e^{-3x} dx$. Of course you know how to find this easy antiderivative by hand. But we'll use it to illustrate how to find integrals using MATLAB.

Type in the following commands:

```
>> x=sym('x');
>> f=exp(-3*x);
>> g=int(f)
ans =
    -1/(3 * exp(3*x))
```

Note that MATLAB has found the antiderivative for e^{-3x} . We know the answer should be $-e^{-3x}/3 + C$, with a "constant of integration" C . MATLAB omits this constant, but that doesn't mean you should!

```
>> diff(g)
ans =
    1/(exp(3*x))
MATLAB has found the derivative of the antiderivative of f, namely f
```

```
>> h = int (f, 0, 1) % f=exp(-3*x)
h =
    1/3-1/(3*exp(3))
MATLAB has found the definite integral of  $e^{-3x}$  from 0 to 1. You may
replace 0 and 1 with any bounds of integration which make sense for
the function you are trying to integrate.
```

```
>> single(h)
ans =
    .3167
MATLAB has found the numerical value of  $-1/3 * \exp(-3) + 1/3$ .
```

Example 2:

```
>> rsums exp(-3*x)
```

Explanation

MATLAB has given you a window which shows the Riemann sums for $\exp(-3*x)$ on the interval $[0, 1]$ when $n = 10$. To increase n , move the scroll button on the horizontal bar on the bottom. The value of the Riemann sum is shown above the graph. You can watch this value change as you increase n .

Advantages of rsums:

it gives you a graph which represents the Riemann sums visually and lets you see how increasing n increases the accuracy of the Riemann approximation.

Example 3:

$\int \sin^7(x) \cos^5(x) dx$. You should know how to evaluate this integral by hand, but the computations are long.

```
>> h = sin(x)^7 * cos(x)^5;
>> k = int(h)
MATLAB gives an answer which is long and messy. Perhaps you can't even see
the whole answer on the screen.
>> simple(k)
The answer is now easier to read.
```

Example 4:

Use MATLAB to compute the derivative of the integral of $\tan^3 x$

```
>> f=tan(x)^3;
>> g=int(f)
g =
    log(cos(x)) - (cos(x)^2 - 1)/(2*cos(x)^2)
>> h=diff(g)
h =
    -(sin(x)*(cos(x)^2 - 1))/cos(x)^3
>> simple(h)
sin(x)/cos(x)^3 - sin(x)/cos(x)
```

(Note that differentiating after integrating doesn't always return us to where we started.)

Exercise 1:

a.) If $f = \tan^4 2x$, then the MATLAB command `diff(f)` equals

(1) **Circle one:**

1. $8*\tan(2*x)^3*\sec(2*x)^2$
2. $4*\tan(2*x)^3*(2*\tan(2*x)^2 + 2)$
3. $4*\tan(2*x)^3*(2+2*\tan(2*x))^2$
4. $\tan(2*x)^2*3*x^2$
5. none of the above

b.) Mathematically, which of the answers in (a) above are equivalent?

(2) **Circle one:** 1. 1 and 2 2. 2 and 3 3. 2 and 4 4. 1 and 3 5. none of the above

2.1 Numerical Approximation, or, What to do When There is no Antiderivative.

We will now give three examples of the kind of answer MATLAB will give if you ask it to find an integral, but the function you give has no antiderivative in terms of elementary functions. We then see how to use MATLAB to find a numerical approximation. This numerical approximation is only for the definite integral.

Example 5:

$$\int \cos(x^3)dx.$$

Type the following:

```
>> x=sym('x');
>> f=cos(x^3);
>> g=int(f)
```

Your answer is `int(cos(x^3), x)`. This mess is not surprising, since the function $\cos x^3$ has no antiderivative in terms of “elementary functions”. MATLAB cannot find an answer by using the Fundamental Theorem of Calculus. We now try a definite integral, by giving the limits of integration 0 and 1:

```
>> h=int(f, 0,1)
h=hypergeom([1/6], [1/2, 7/6], -1/4)
```

Again, we get an unhelpful result. The answer can’t be obtained by finding an antiderivative.

Now type:

```
>> single(h)
ans=
    0.9317
```

EXPLANATION: MATLAB has found a numeric approximation to $\int_0^1 \cos(x^3)dx$. You can change the limits of integration to whatever makes sense in your problem. But it is necessary to give some limits of integration.

2.2 Practice finding integrals.

DIRECTIONS:

You know how to find antiderivatives for some of the functions below, but not for all of them.

- (a) In the first part of each of the following exercises indicate the method used to find the antiderivative. Select “MATLAB” as an answer to this first part only if none of the integration techniques work, that is – Substitution, Integration by Parts or by Trigonometric functions.

- (b) Sometimes the right answer is, “Explicit integral cannot be found”

Exercise 2:

- a.) $\int \ln x dx$. Which of the integration techniques should you use to find the antiderivative?

(3) Circle one:

- 1. Trigonometric substitution
- 2. Substitution (reverse of the chain rule)
- 3. Integration by parts
- 4. MATLAB
- 5. Explicit integral cannot be found

b.) $\int \ln x dx$ equals

(4) Circle one:

1. $\ln(x)^2/2$ 2. $1/x$ 3. $x \ln x - x$ 4. $\ln(x)^2/x$ 5. none of the above

Exercise 3:

a.) $\int \frac{1}{\sqrt{1-4x^2}} dx$. Which of the integration techniques should you use to find the antiderivative?

(5) Circle one:

1. Trigonometric substitution 2. Substitution (reverse of the chain rule) 3. Integration by parts 4. MATLAB 5. Explicit integral cannot be found

b.) $\int \frac{1}{\sqrt{1-4x^2}} dx$ equals

(6) Circle one:

1. $1/2 \sinh^{-1} 2x$ 2. $1/2 \arcsin 2x$ 3. $2/3(1+x^2)^{3/2}$ 4. $(1+x^2)^{1/2}$ 5. none of the above

Exercise 4:

a.) $\int x e^{x^2} dx$. Which of the integration techniques should you use to find the antiderivative?

(7) Circle one:

1. Trigonometric substitution 2. Substitution (reverse of the chain rule) 3. Integration by parts 4. MATLAB 5. Explicit integral cannot be found

b.) $\int x e^{x^2} dx$ equals

(8) Circle one:

1. $x e^{x^2}$ 2. $2x e^{x^2}$ 3. e^{x^2} 4. $1/2 e^{x^2}$ 5. None of the above

Exercise 5:

a.) $\int \frac{x}{\sqrt{1+x^2}} dx$. Which of the integration techniques should you use to find the antiderivative?

(9) Circle one:

1. Trigonometric substitution 2. Substitution (reverse of the chain rule) 3. Integration by parts 4. MATLAB 5. Explicit integral cannot be found

b.) $\int \frac{x}{\sqrt{1+x^2}} dx$ equals

(10) Circle one:

1. $(1+x^2)^{1/2}$ 2. $\sinh^{-1} x$ 3. $2/3(1+x^2)^{3/2}$ 4. $\arcsin x$ 5. none of the above

Exercise 6:

a.) $\int \frac{1}{\sqrt{1+x^2}} dx$. Which of the integration techniques should you use to find the antiderivative?

(11) Circle one:

1. Trigonometric substitution 2. Substitution (reverse of the chain rule) 3. Integration by parts 4. MATLAB 5. Explicit integral cannot be found

b.) $\int \frac{1}{\sqrt{1+x^2}} dx$ equals

(12) Circle one:

1. $2/3(1+x^2)^{3/2}$ 2. $\sinh^{-1}x$ or $\ln |(1+x^2)^{1/2} + x|$ 3. $\arcsin x$ 4. $(1+x^2)^{1/2}$ 5. none of the above

Exercise 7:

a.) $\int x^3 \sin x dx$. Which of the integration techniques should you use to find the antiderivative?

(13) Circle one:

1. Trigonometric substitution 2. Substitution (reverse of the chain rule) 3. Integration by parts 4. MATLAB 5. Explicit integral cannot be found

b.) $\int x^3 \sin x dx$ equals

(14) Circle one:

1. $-1/3 \sin^2 x \cos(x) - 2/3 \cos x$ 2. $-x^4/4 \cos x$ 3. $-x^3 \cos x + 3x^2 \sin x - 6 \sin x + 6x \cos x$ 4. $x^3 \cos x - 3x^2 \sin x + 6 \sin x - 6x \cos x$ 5. none of the above

Exercise 8:

a.) $\int \sin^3 x dx$. Which of the integration techniques should you use to find the antiderivative?

(15) Circle one:

1. Trigonometric substitution 2. Substitution (reverse of the chain rule) after replacing trig function 3. Integration by parts 4. MATLAB 5. Explicit integral cannot be found

b.) $\int \sin^3 x dx$ equals

(16) Circle one:

1. $-1/3 \sin^2 x \cos(x) - 2/3 \cos x$ 2. $-x^4/4 \cos x$ 3. $-x^3 \cos x + 3x^2 \sin x - 6 \sin x + 6x \cos x$ 4. $x^3 \cos x - 3x^2 \sin x + 6 \sin x - 6x \cos x$ 5. none of the above

Exercise 9:

- a.) $\int \sin(x^3)dx$. Which of the integration techniques should you use to find the antiderivative?
(17) Circle one: 1. Substitution (reverse of the chain rule) 2. Integration by parts 3. Trigonometric substitution 4. Explicit integral cannot be found – MATLAB estimates it in terms of the “hypergeom” function
- b.) $\int \sin(x^3)dx$ equals
(18) Circle one: 1. $-1/3 \sin^2 x \cos(x) - 2/3 \cos x$ 2. $-x^4/4 \cos x$ 3. $-x^3 \cos x + 3x^2 \sin x - 6 \sin x + 6x \cos x$ 4. $x^3 \cos x - 3x^2 \sin x + 6 \sin x - 6x \cos x$ 5. none of the above

Exercise 10:

- a.) $\int_0^\pi \sin(x^3)dx$. Which of the integration techniques should you use to evaluate the definite integral?
(19) Circle one:
1. Substitution (reverse of the chain rule) 2. Integration by parts 3. Trigonometric substitution 4. Explicit integral cannot be found – MATLAB estimates it in terms of the “hypergeom” function
- b.) $\int_0^\pi \sin(x^3)dx \approx$ **(20) Circle one:**
1. 0.4158 2. 1.3333 3. 0.4999 4. 12.1567 5. none of the above

MTH232

The College of Staten Island
Department of Mathematics

Taylor Polynomials and Approximations

New MATLAB Commands:

taylor(f,n)
prism
subs

1 Definition of the Taylor Polynomial

The n th Taylor polynomial which approximates a function $f(x)$ near a point c is given by:

$$P_n(x) = f(c) + f'(c)(x - c) + \frac{f''(c)}{2!}(x - c)^2 + \dots + \frac{f^{(n)}(c)}{n!}(x - c)^n$$

The n th Taylor polynomial has many of the properties of $f(x)$, i.e. $p_n(c) = f(c)$, the slopes of both $p_n(x)$ and $f(x)$ agree at c , both functions have the same concavity at c (assuming $n \geq 2$). Indeed, both $p_n(x)$ and $f(x)$ agree on all the mathematical properties controlled by the first n derivatives at c . However, they are not equal. There is an error term in using a polynomial to approximate a function which might not be a polynomial. Taylor's Theorem states that

$$f(x) = p_n(x) + R_n(x) \quad \text{where} \quad R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!}(x - c)^{n+1}$$

and z lies between x and c .

1.1 Objective

The object of this project is to use the sequence of Taylor polynomials $\{p_0(x), p_1(x), p_2(x), \dots, p_n(x), \dots\}$ to approximate $f(x)$ near c by graphing the Taylor polynomials and $f(x)$ as well as by determining numerically the error in using the Taylor polynomials as an approximation.

2 Using Taylor Polynomials to Approximate $\cos x$

Exercise 1:

We want to calculate some of the Taylor polynomials centered at $c = 0$ for $\cos x$, i.e. MacLaurin polynomials for $\cos x$, and we want to graph these functions on the same graph with $\cos(x)$ in order to view the error.

a.) Using the formulas above, compute $p_0(x)$, $p_1(x)$, $p_2(x)$, $p_4(x)$, for $f(x) = \cos(x)$ centered at $c = 0$. (You can do this without MATLAB)

- $p_0(x) =$
(1) Circle one:
 1. $P_0(x) = 0$
 2. $P_0(x) = 1 - x^2/2$
 3. $P_0(x) = 1$
 4. $P_0(x) = 1 - x^2/2 + x^4/24$

- $P_1(x) =$
(2) Circle one:
 1. $P_1(x) = 0$
 2. $P_1(x) = 1 - x^2/2$
 3. $P_1(x) = 1$
 4. $P_1(x) = 1 - x^2/2 + x^4/24$

- $P_2(x) =$
(3) Circle one:
 1. $P_2(x) = 0$
 2. $P_2(x) = 1 - x^2/2$
 3. $P_2(x) = 1$
 4. $P_2(x) = 1 - x^2/2 + x^4/24$

- $P_4(x) =$
(4) Circle one:
 1. $P_4(x) = 0$
 2. $P_4(x) = 1 - x^2/2$
 3. $P_4(x) = 1$
 4. $P_4(x) = 1 - x^2/2 + x^4/24$

2.1 Plot the Graphs of $\cos(x)$, $P_2(x)$ and $P_4(x)$

Example 1:

The commands below plot the graphs of $\cos(x)$, $P_2(x)$ and $P_4(x)$

We can use the command `ezplot` which allows you to graph a function $f(x)$ on an interval $[-\pi, \pi]$.

```
>> syms x
>> ezplot(cos(x), [-pi, pi]), grid
>> hold on                                % previous and subsequent graphs will appear on
                                         % same window
>> ezplot(1-x^2/2+x^4/24, [-pi, pi])      % the  $P_4(x)$  you calculated above
>> ezplot(1-x^2/2, [-pi, pi])            % the  $P_2(x)$  you calculated above
>> prism                                  % This will make each plot a different color, so you
                                         % can easily tell them apart
```

2.2 Determining Error

Exercise 2:

Look at the graph of the functions of $\cos(x)$ and $P_4(x)$.

- a.) By looking at the graph that you generated above using MATLAB and using `zoom on`, estimate the numerical error (the distance) between the points $(\pi, \cos(\pi))$ and $(\pi, P_4(\pi))$?

(5) Answer: _____

2.2.1 Subs command

In Project 2 you used the MATLAB command `subs` which lets you symbolically substitute a number into a function expression. You can then use MATLAB to calculate the numerical value.

```
>> syms f x
>> f=cos(x)
>> subs(f, pi)
ans =    -1
```

Exercise 3:

- a.) Use MATLAB to calculate the numerical error if you tried to approximate $\cos(\pi)$ by $p_4(\pi)$?

(6) Answer: _____

- b.) Taylor's theorem says that the error $|R_4(\pi)|$ must be less than or equal to $|f^{(5)}(z)|/5!(\pi)^5$ for some value of z between 0 and π . What is the largest this bound on the error could be?

(7) Answer: _____

3 Using MATLAB to Symbolically Calculate Taylor Polynomials

The Maple Toolbox has a command `taylor(f,n)` which symbolically calculates the n^{th} Taylor expansion for the function f

Note: The command `taylor(f,n)` returns the first n terms of the Taylor expansion centered at $c = 0$ (called the Maclaurin expansion).

Example 2:

For the function $\ln(1+x)$ the MATLAB expression is: `f = log(1+x)`
`taylor(f,6)` returns....

$$x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5$$

To get the Taylor Polynomial of Degree 5:

```
>> syms f x p5
>> f=log(1+x)
>> p5=taylor(f,6)
p5 = x - 1/2*x^2 + 1/3*x^3 - 1/4*x^4 + 1/5*x^5 % p5 is shorthand for P5(x)
```

$$p_5 = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5$$

To plot the graph and get a good view near $c = 0$:

```
>> hold off
>> ezplot(f)
>> hold on
>> ezplot(p5)
>> axis([-pi/2 pi/2 -5 5])
>> prism
>> grid
```

Exercise 4:

Using the commands as above, generate the 7^{th} Taylor polynomial for the function $f = \tan x$ and answer the questions below:

- a.) Look at the graph of the functions $\tan(x)$ and $P_7(x)$.
How good an approximation is P_7 for $\tan(x)$ on the interval $[-\pi, \pi]$?

(8) Circle all that apply:

1. the approximation is poor when x is near $\pi/2$
2. the approximation is good when x is near zero
3. the approximation is good when x is near $\pi/2$
4. the approximation is poor when x is near zero

b.) Using MATLAB, find $P_7(\pi/4)$

(9) Answer: _____

c.) Find $\tan(\pi/4)$

(10) Answer: _____

d.) Using MATLAB, find $P_7(\pi/6)$

(11) Answer: _____

e.) Find $\tan(\pi/6)$

(12) Answer: _____

f.) The errors between $\tan x$ and $P_7(x)$ will be calculated at $\pi/4$ and $\pi/6$:

- Compute the error between $\tan(\pi/4)$ and its approximation, $P_7(\pi/4)$.

(13) Answer: _____

- Compute the error between $\tan(\pi/6)$ and its approximation, $P_7(\pi/6)$.

(14) Answer: _____

- With respect to the magnitude of the errors, is this what you would have expected?

(15) Circle one:

1. no, the errors should be the same 2. no, the error should be greater near $\pi/6$ 3. none of the above 4. yes, the error is smaller near $c = 0$

Exercise 5:

We will use MATLAB to approximate the function $f(x) = 2x^2 + \cos(3x)$.

- First, make a graph of this function on the interval $[-1, 1]$.
- Then use the `taylor` command to compute $p3(x)$ and $p5(x)$ for $f(x) = 2x^2 + \cos(3x)$.
- Plot these functions making sure to use the “hold on” and “prism” commands. Notice that as you get far away from $c = 0$, the approximations are not as good.

- Label each approximation using the `gtext` command:

```
>> gtext('whatever you want as your label')
```

For example, to label the 3th approximation, type:

```
>> gtext('p3')
```

Then place your mouse pointer at the point in the graph window where you want this label to appear and click on the mouse. The label will be printed in the location it appears in the graphing window.

(Alternately, you can use the graphical interface provided in the figure window.)

“Taylor Polynomials and Approximations”

Consider the following: How many terms do you need to have in the Taylor approximation to f in order to get a polynomial which agrees with f pretty closely on the entire interval $[-1, 1]$? Let's say that we want to find a Taylor approximation which has an error of at most 0.1 from the function f . To find out how many terms are needed, plot successively better and better approximations. For example, you might compute p_6 , p_6 , p_7 or even better approximations. Hint: Look at the endpoints.

You will soon find an approximation that is within 0.1 of the function f . Use the `zoom` command to tell how good your current approximation is, to determine if you need to make a better approximation.

- a.) How many terms of the Taylor series did you have to include to find a Taylor approximation which was within, say, an error of 0.1 from the function f ?

(16) Answer:

- b.) Submit the graphing window when it contains $f(x)$ and $p_5(x)$ and the Taylor approximation that is within 0.1 of the function f . Label the graphs. (17) **Attach your graph to the worksheet.**

MTH232

The College of Staten Island
Department of Mathematics

Polar Graphs

MATLAB Commands:

```
polar(t,r)
[x,y]=pol2cart(t,r)
comet(x,y)
```

We can easily draw polar graphs by using MATLAB’s “**polar(t,r)**” command. This command makes a plot by using polar coordinates of the angle t (expressed in radians) versus the radius r .

Example 1:

Plot $r = \sin(5t)$ and then determine what interval for t is needed in order to trace the entire graph only *once*.

```
>> t=0:pi/180:3*pi;
>> r=sin(5*t);
>> polar(t,r)
```

The graph is shown below.

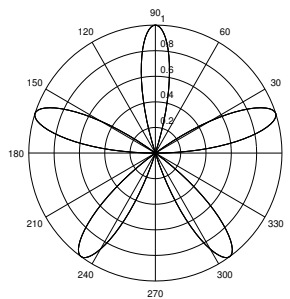


Figure 1: Graph of $r = \sin 5t$

By repeating the following commands, we can experiment with different values of t until we determine the interval required to trace the entire graph only once.

```
>> t=0:pi/180:3*pi/4;           % experiment by changing the "3*pi/4"
>> r=sin(5*t);
>> polar(t,r)
```

Exercise 1:

Use MATLAB to plot the graphs of each of the following. Then determine what interval for t is needed in order to trace the entire graph only once. (Use subplot(2,2,1) through subplot(2,2,3) to get the three graphs onto one window.)

a.) interval for t in order to trace $r = 4 \cos(2t)$ only once:

(1) Circle one:

1. $[0, \pi/3]$ 2. $[0, \pi/2]$ 3. $[0, 2\pi]$ 4. $[0, \pi]$

b.) interval for t in order to trace $r = \cos(5t)$ only once:

(2) Circle one:

1. $[0, \pi/3]$ 2. $[0, \pi/2]$ 3. $[0, 2\pi]$ 4. $[0, \pi]$

c.) interval for t in order to trace $r = \sin(t/2)$ only once:

(3) Circle one:

1. $[0, 4\pi]$ 2. $[0, 3\pi]$ 3. $[0, 2\pi]$ 4. $[0, \pi]$

d.) Submit a print-out of your graphs

(4) Attach your graph to the worksheet.

Exercise 2:

a.) Use MATLAB to plot $r = \sin(2t)$ and $\cos(2t)$ on the same graph.

(5) Attach your graph to the worksheet.

b.) $\sin(2t) =$

(6) Circle one:

1. $\cos(2t - \pi/3)$ 2. $\cos(2t - \pi/4)$ 3. $\cos(2t + \pi/4)$ 4. $\cos(2t - \pi/2)$

Exercise 3:

a.) Use MATLAB to draw the graph of $r = 6 - 4 \sin(t)$. Submit the graph

(7) Attach your graph to the worksheet.

b.) $r = 6 - 4 \sin(t)$ is a

(8) Circle one:

1. rose 2. limaçon 3. circle 4. cardioid

The MATLAB command `[x,y] = pol2cart(t,r)` converts data stored in polar coordinates to cartesian coordinates. You can then plot these cartesian points using “`plot(x,y)`” or you can use the “`comet(x,y)`” command to draw the graph in slow motion. With `comet`, you can observe the direction the petals take as t increases. This gives a helpful frame of reference when trying to compute area between curves.

Example 2:

Find the area within 4 petals common to $r = 4 \sin(2t)$ and $r = 2$. You should verify that the points of intersection between $t = 0$ to $\pi/2$ are $t = \pi/12$ and $5\pi/12$. We will use MATLAB to see the graph more clearly.

```
>> t=0:pi/180:2*pi;
>> r1=4*sin(2*t);
>> r2=t.^0+1;           % Note that t.^0 equals 1.
>> polar(t,r1)
>> hold on
>> polar(t,r2,'g')
```

The graphs are shown below. We note that the graph is symmetrical with respect to $t = 0$ and $t = \pi/2$. So we find the area between $t = 0$ and $t = \pi/2$ and multiply by four to get the entire area.

Note: The graph is also symmetrical with respect to $t = 0$ and $t = \pi/4$, and if you used this symmetry you would need to multiply by 8 to get the entire area.

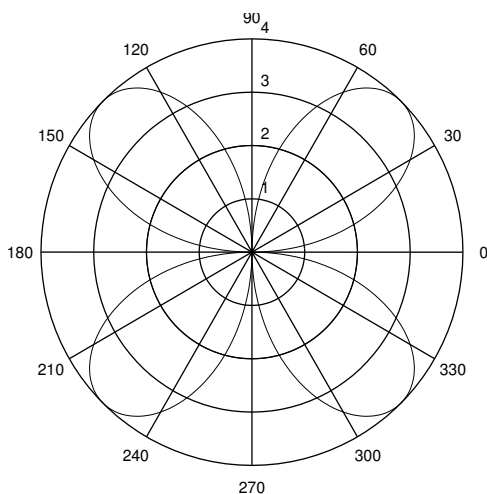


Figure 2: Graph of $r = 4 \sin 2t$

$$Area = 4 \left[\frac{1}{2} \int_0^{\pi/12} (4 \sin 2t)^2 dt + \frac{1}{2} \int_{\pi/12}^{5\pi/12} (2)^2 dt + \frac{1}{2} \int_{5\pi/12}^{\pi/2} (4 \sin 2t)^2 dt \right]$$

Verify that the area = $4\sqrt{3} + \frac{16\pi}{3} = 9.8270$

```
>> syms t
>> f=4*sin(2*t)
>> 4*(1/2*int(f^2,t,0,pi/12)+1/2*int(2^2,t,pi/12,5*pi/12)+1/2*int(f^2,t,5*pi/12,pi/2))
ans = -4*3^(1/2)+16/3*pi
>> single(ans)
ans = 9.8270
```

Exercise 4:

a.) Find the point of intersection for $r = 8 \cos^2(2t)$ and $r = 4$ where $0 < t < \pi/4$

(9) Circle one:

1. $\pi/8$ 2. $\pi/6$ 3. $\pi/10$ 4. $\pi/12$

b. Find the area within 4 petals common to $r = 8 \cos^2 2t$ and $r = 4$.

(10) Circle one:

1. $20\pi + 32$ 2. 16π 3. $\pi/4$ 4. $20\pi - 32$

